

# GENERALIZATIONS OF THE KUNEN INCONSISTENCY

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**ABSTRACT.** We present several generalizations of the well-known Kunen inconsistency that there is no nontrivial elementary embedding from the set-theoretic universe  $V$  to itself. For example, there is no elementary embedding from the universe  $V$  to a set-forcing extension  $V[G]$ , or conversely from  $V[G]$  to  $V$ , or more generally from one set-forcing ground model of the universe to another, or between any two models that are eventually stationary correct, or from  $V$  to  $\text{HOD}$ , or conversely from  $\text{HOD}$  to  $V$ , or indeed from any definable class to  $V$ , among many other possibilities we consider, including generic embeddings, definable embeddings and results not requiring the axiom of choice. We have aimed in this article for a unified presentation that weaves together some previously known unpublished or folklore results, several due to Woodin and others, along with our new contributions.

The Kunen inconsistency [Kun71], the theorem showing that there can be no nontrivial elementary embedding from the universe to itself, remains a focal point of large cardinal set theory, marking a hard upper bound at the summit of the main ascent of the large cardinal hierarchy, the first outright refutation of a large cardinal axiom. On this main ascent, large cardinal axioms assert the existence of elementary embeddings  $j : V \rightarrow M$  where  $M$  exhibits increasing affinity with  $V$  as one climbs the hierarchy. The  $\theta$ -strong cardinals, for example, have  $V_\theta \subseteq M$ ; the  $\lambda$ -supercompact cardinals have  $M^\lambda \subseteq M$ ; and the huge cardinals have  $M^{j(\kappa)} \subseteq M$ . The natural limit of this trend, first suggested by Reinhardt, is a nontrivial elementary embedding  $j : V \rightarrow V$ , the critical point of which is accordingly known as a *Reinhardt* cardinal. Shortly after this idea was introduced, however, Kunen famously proved using the axiom of choice that there are no such embeddings and hence no Reinhardt cardinals.

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**Theorem 1** (The Kunen Inconsistency). *There is no nontrivial elementary embedding  $j : V \rightarrow V$ .*

In this article, we present several generalizations of this theorem, thereby continuing what has been a small industry of generalizations of this central result, including Harada [Kan04, p. 320-321], Woodin [Kan04, p. 322], Zapletal [Zap96] and Suzuki [Suz98, Suz99]. In order to emphasize a coherent theme of mathematical ideas, we give a unified presentation that includes some previously known generalizations and unpublished folklore results along with our new contributions. A fitting alternative title for our paper might therefore have been “Generalizations of generalizations of the Kunen inconsistency.” Among the generalizations of the Kunen inconsistency we establish in this article, several due to Woodin and others, are the following:

- (1) There is no nontrivial elementary embedding  $j : V[G] \rightarrow V$  of a set-forcing extension of the universe to the universe, and neither is there  $j : V \rightarrow V[G]$  in the converse direction.
- (2) More generally, there is no nontrivial elementary embedding between two set-forcing ground models of the universe.
- (3) More generally still, there is no nontrivial elementary embedding  $j : M \rightarrow N$  when both  $M$  and  $N$  are eventually stationary correct.
- (4) There is no nontrivial elementary embedding  $j : V \rightarrow \text{HOD}$ , and neither is there  $j : V \rightarrow M$  for a variety of other definable classes, including  $\text{gHOD}$  and the  $\text{HOD}^\eta$ ,  $\text{gHOD}^\eta$ .
- (5) If  $j : V \rightarrow M$  is elementary, then  $V = \text{HOD}(M)$ .
- (6) There is no nontrivial elementary embedding  $j : \text{HOD} \rightarrow V$ .
- (7) More generally, for any definable class  $M$ , there is no nontrivial elementary embedding  $j : M \rightarrow V$ .
- (8) There is no nontrivial elementary embedding  $j : \text{HOD} \rightarrow \text{HOD}$  that is definable in  $V$  from parameters.

This list is just a selection; all the details and additional more refined generalizations appear in the subsequent theorems of this article, including other natural definable classes, such as the iterated  $\text{HOD}^\eta$ , the generic- $\text{HOD}$  and  $\text{gHOD}^\eta$ , generic embeddings, definable embeddings and results not requiring the axiom of choice.

## 1. A FEW METAMATHEMATICAL PRELIMINARIES

Before getting to the actual generalizations of the Kunen inconsistency, let us begin by dispelling a few metamathematical clouds that occasionally obscure the large cardinal summit of the Kunen inconsistency. We should like briefly to clarify these metamathematical issues.

The first concerns the fact that the Kunen inconsistency is explicitly a second-order claim, as the purported embedding  $j$  that it rules out would clearly be a proper class of some kind. In particular, the statement of the Kunen inconsistency does not seem directly to be expressible in the usual first-order language of set theory, as it quantifies over second-order objects: “*There is no  $j$  such that...*” So how and in which theory shall we take it as a precise mathematical claim?

To be sure, many large cardinal notions are characterized by second-order assertions that turn out to have first-order equivalent formulations, which can be treated in ZFC. For example, a cardinal  $\kappa$  is measurable if it is the critical point of an elementary embedding of  $V$  into a transitive class, and this is equivalent to the first-order assertion that there is a nonprincipal  $\kappa$ -complete ultrafilter on  $\kappa$ . But it is easy to see that there can be no corresponding consistent first-order formulation of Reinhardt cardinals, since if  $\kappa$  is the least Reinhardt cardinal, then by elementarity  $j(\kappa)$  will also be least, an immediate contradiction. More generally, there can be no consistent first-order property  $\varphi(\kappa)$  that implies that  $\kappa$  is Reinhardt, because if  $\kappa$  is the least cardinal with that property, then by elementarity so is  $j(\kappa)$ , again a contradiction. So there is no completely first-order account of the Reinhardt cardinal concept, and the issue is how then we are to formalize Reinhardt cardinals and the Kunen inconsistency statement in some second-order manner. Although there are a variety of satisfactory resolutions of this issue, aligning with the various treatments of classes and proper classes that are available in set theory, it turns out that they are not equally efficacious, for some provide a greater substance for the Kunen inconsistency than others.

One traditional approach to classes in set theory, working purely in ZFC, is to understand all talk of classes as a substitute for the first-order definitions that might define them. In this formulation, the Kunen inconsistency becomes a theorem scheme, asserting that no formula defines (with parameters) a nontrivial elementary embedding of the universe to itself. Thus, for each first-order formula  $\psi$  in the language of set theory, we have the theorem that for no parameter  $z$  does the relation  $\psi(x, y, z)$  define a function  $y = j(x)$  that is an elementary embedding from  $V$  to  $V$ . This is the approach used in [Kan97, Kan04].<sup>1</sup>

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<sup>1</sup>“As the quantification  $\forall j$  over classes  $j$  cannot be formalized in ZFC, this result can only be regarded as a schema of theorems, one for each  $j$ ” [Kan04, p. 319], see also [Kan97, p. 319]. The author also notes, however, that the purely first-order strengthening of the theorem to the assertion that there is no nontrivial  $j : V_{\lambda+2} \rightarrow V_{\lambda+2}$  is expressible and provable purely in ZFC.

Our view is that this way of understanding the Kunen inconsistency does not convey the full power of the theorem. Part of our reason for this view is that if one is concerned only with such definable embeddings  $j$  in the Kunen inconsistency, then in fact there is a far easier proof of the result, simpler than any of the traditional proofs of it and making no appeal to any infinite combinatorics or indeed even to the axiom of choice. We explain this argument in theorem 32.

Instead, a fuller power for the Kunen inconsistency seems to be revealed when it is understood as a claim in a true second-order set theory, such as von Neumann-Gödel-Bernays set theory NGBC (see [Jec03, p. 70]; this theory is also commonly known as Gödel-Bernays set theory) or Kelley-Morse KM set theory. Kunen himself understood his result to be formalized in KM, writing:

It is intended that our results be formalized within the second order Morse-Kelley set theory (as in the appendix to Kelley [Kel65]), so that statements involving the satisfaction predicate for class models can be expressed. ([Kun71, p. 407])

Meanwhile, as we shall explain below, it turns out that for the purposes of the Kunen inconsistency, one can sufficiently express the satisfaction relation and prove the theorem in the strictly weaker theory NGBC, even in the subtheory  $\text{NGB} + \text{AC}$ , or in the fragment of this we denote by  $\text{ZFC}(j)$  below.

In these second-order theories, one distinguishes between the first-order objects, the sets, and the second-order objects, the classes, (although there are elegant economical accounts that unify the treatment purely in terms of classes). The crucial difference between the theories is that NGB includes the replacement and separation axioms only for formulas having only first-order quantifiers, that is, quantifying only over sets, allowing finitely many class parameters, whereas KM allows formulas into the schemes that quantify also over classes. The theories NGBC and KM include also a global choice principle, whereas the theory NGB omits any choice principle.

The NGBC theory is conservative over ZFC, meaning that any first-order assertion about sets provable in NGBC is also provable in ZFC, and this can be easily proved by expanding any model of ZFC to a model of NGBC by adding a generic global choice class, if necessary, and then interpreting the second-order part to consist precisely of the classes that are definable from this class and set parameters. In particular, NGBC and ZFC are equiconsistent. The theory KM, in contrast, is strictly stronger than ZFC in consistency strength (if consistent),

because it proves that there is a satisfaction predicate for first-order truth, and indeed, that there is a satisfaction predicate for first-order truth relativized to any class parameter. So KM proves  $\text{Con}(\text{ZFC})$  and  $\text{Con}(\text{ZFC} + \text{Con}(\text{ZFC}))$  and so on transfinitely, and therefore is not conservative over ZFC, if this theory is consistent.

In some of our arguments below, we will use the forcing method in the NGBC context, and so let us remark without elaboration that the usual theory of forcing goes through fine in this second-order setting: the classes of the forcing extension are obtained from the classes of the ground model by interpreting them in the usual name fashion, NGBC is preserved and all the usual forcing technology works as expected. We say that  $M$  is a ground model of  $N$  if the latter can be realized as a forcing extension  $N = M[G]$  for some  $M$ -generic filter  $G \subseteq \mathbb{P} \in M$ . In order to emphasize that the forcing should be a set in  $M$ , we will sometimes say that  $M$  is a set-forcing ground model.

Proving the Kunen inconsistency in NGBC appears to give a stronger result than either the ZFC scheme approach or the KM approach, the former because it rules out not only the definable embeddings, but also the possibility that a non-definable embedding  $j : V \rightarrow V$  may arise as a class in an NGBC model, and the latter simply because NGBC is a weaker theory than KM and closer to ZFC. The easy proof of the definable embedding version of the Kunen inconsistency, as in theorem 32, does not seem to generalize to establish the stronger NGBC result. Therefore, for the rest of this article, unless otherwise stated, we shall work formally in NGBC set theory. But actually, none of our arguments needs the global version of choice, and so  $\text{NGB} + \text{AC}$  suffices for us; meanwhile, when we say below, “Do not assume AC,” we intend to work in NGB. As the class  $j$  and the ones definable from it will be the only classes we need to consider, we could alternatively formalize the presentation of our theorems in the theory  $\text{ZFC}(j)$ , the intermediate subtheory of  $\text{NGB} + \text{AC}$  where one has fixed a single class predicate for  $j$ , which is allowed to appear in the formulas of the replacement and separation axiom schemes, or in  $\text{ZF}(j)$  when we do not use AC. When  $M$  is a model of ZFC, we shall say that  $j$  is a class of  $M$  to mean that  $(M, j)$  is a model of  $\text{ZFC}(j)$ .

The second metamathematical issue concerning the Kunen inconsistency that we would like to discuss, which arises whether one uses the ZFC theorem scheme approach, the NGBC approach or the KM approach, is that the theorem involves the hypothesis that  $j$  is an elementary embedding, and it is not immediately clear how to express such a hypothesis in our language. Naïvely, the assertion that  $j : V \rightarrow V$  is elementary is expressed most plainly by a scheme of statements, those

of the form  $\forall x [\varphi(\vec{x}) \longleftrightarrow \varphi(j(\vec{x}))]$ , rather than by a single statement. But a scheme does not seem to serve the purpose here, because the assertion that  $j$  is elementary appears negatively in the theorem—either as the antecedent of an implication or, in the contrapositive, as the claim ultimately that  $j$  is not elementary—whereas the negation of a scheme is not generally expressible even as a scheme. So we cannot seem to use a scheme account of elementarity in order to find a coherent statement of the theorem, even as a scheme, and even when  $j$  itself is assumed to be defined by a given formula  $\psi$ . So again, how are we precisely to express the theorem?

Kunen observed that this issue is addressed in KM set theory by the fact that KM proves the existence of a class satisfaction predicate for firstorder truth, by means of which the elementarity of  $j$  is expressible. Meanwhile, at around the same time as the Kunen inconsistency, set-theorists realized how to express the elementarity of  $j$  in the weaker theory NGB by making use the observation that every  $\Delta_0$ -elementary cofinal embedding of models of ZF is fully elementary. An embedding  $j : M \rightarrow N$  of transitive classes is *cofinal* if for every  $y \in N$  there is  $x \in M$  with  $y \in j(x)$ . Equivalently,  $N = \bigcup j'' M$ .

**Lemma 2** (Gaifman [Gai74]). *If  $j : M \rightarrow N$  is  $\Delta_0$ -elementary and cofinal, where  $M$  satisfies ZF, then  $j$  is fully elementary.*

The conclusion of the lemma, that  $j$  is fully elementary, is expressed as a scheme, consisting of the assertions  $\forall x \in M [\varphi^M(x) \iff \varphi^N(j(x))]$  for every formula  $\varphi$ , and so the lemma is technically a lemma scheme, asserting that any such embedding has any desired degree of elementarity. One may regard the lemma instead as the scheme asserting for each natural number  $n$  that  $j$  is  $\Sigma_n$ -elementary, which we may prove by meta-theoretic induction on  $n$ . The atomic and Boolean combination cases are easy, as is the forward direction of the extensional case. The backward direction uses the cofinality hypothesis to transform an unbounded existential to a bounded existential, which reduces to the inductive hypothesis. This final step of the argument is easy when one assumes that both  $M$  and  $N$  satisfy ZF, since one can appeal to the absorption of bounded quantification by  $\Sigma_n$  assertions in a ZF-provably canonical manner. Gaifman's observation was that, in fact, it can be carried out even if one assumes only that  $M$  satisfies ZF, or even less. For example, [GHJ] provides full details for the context of  $\text{ZFC}^-$ , set theory without the power set axiom. Also, [Kan04, p. 45] has some further details, including information about the fact that if  $M$  and  $N$  are transitive proper class models of ZF containing all the ordinals and  $j : M \rightarrow N$  is  $\Sigma_1$ -elementary, then in fact the cofinality assumption of

lemma 2 follows for free. This is because  $j(V_\alpha^M) = V_{j(\alpha)}^N$ , as the relation “ $x = V_\alpha$ ” has complexity  $\Pi_1$ , and since  $\alpha \leq j(\alpha)$ , these sets union up to  $N$ , which implies that  $j$  is cofinal.

One important difference between the NGBC approach to expressing the elementarity of  $j$  via Gaifman’s theorem and the KM approach using a satisfaction predicate is that in the former account, one derives any desired instance of elementarity by meta-theoretic induction from the assumption that  $j$  is  $\Delta_0$  elementary and cofinal, whereas in KM one uses the internal assertion that  $j$  is elementary with respect to the first-order satisfaction class that KM proves to exist. In particular, in the KM context, such embeddings are also elementary with respect to the possibly nonstandard formulas, and one can imagine inductive arguments that rely internally on  $\Sigma_n$ -elementarity for every natural number  $n$ , a possibility that it seems could not be carried out in NGBC approach via Gaifman’s theorem, because in that approach one has such full elementarity only as a meta-theoretic scheme.

Note that if  $M$  is an NGBC model and  $j : M \rightarrow N$  is elementary in the sense of lemma 2, then we may extend the domain of  $j$  to include every NGBC class  $A$  (including in particular  $A = j$ , if this should be a class of  $M$ ) by defining  $j(A) = \bigcup_{\alpha \in \text{ORD}} j(A \cap V_\alpha)$ . This extended embedding remains cofinal and  $\Delta_0$ -elementary from the structure  $\langle M, \in, A \rangle \rightarrow \langle N, \in, j(A) \rangle$ , and so by the argument of Gaifman’s lemma, it is fully elementary in the expanded context, meaning that for any set  $x$  and class  $A$  we have  $M \models \varphi(x, A) \iff N \models \varphi(j(x), j(A))$  for any formula  $\varphi$  with only first-order quantifiers. Since we have used  $j(A)$  on the right, this is slightly weaker from saying that  $j$  is elementary in the language with  $A$ , if one should take this phrase to mean the stronger property that  $A$  is interpreted the same in both  $M$  and  $N$ . In particular, we rarely expect an embedding  $j$  to be elementary in the language with  $j$  itself, if  $j$  is interpreted as  $j$  in both the domain and the range, since the critical point  $\kappa = \text{cp}(j)$  is definable from  $j$ , but it is not in the range of  $j$ , which would contradict this stronger form of elementarity with respect to  $j$ . Rather, if  $j : M \rightarrow N$  and  $j$  is a class of  $M$ , then what we have is elementarity  $j : \langle M, \in, j \rangle \rightarrow \langle N, \in, j(j) \rangle$ , using  $j(j)$  on the right.

A third metamathematical issue, which does not arise for the Kunen inconsistency itself, but which does arise for several of the generalizations of it, is the question of how to express that a given transitive proper class  $M$  is in fact a model of ZF or of ZFC. After all, a naïve formalization of this would seem to involve a scheme of assertions, asserting that  $M \models \psi$  for every axiom  $\psi$ . Nevertheless, it is well-known

that for a transitive proper class one can reduce this entire scheme to a single first-order assertion about  $M$  as described in lemma 3 below. A transitive class  $M$  is *almost universal* if for every set  $x$  such that  $x \subseteq M$ , there exists  $y \in M$  such that  $x \subseteq y$ .

**Lemma 3** ([Jec03, Theorem 13.9]). *A transitive proper class  $M$  is a model of ZF if and only if it is closed under the finitely many Gödel operations and is almost universal. This property of  $M$  is expressible in a single first-order assertion using class parameter  $M$ .*

In light of all these observations, let us adopt the following conventions as a matter of definition. For a transitive proper class  $M$ , by the phrase “ $M$  satisfies ZF,” we mean that  $M$  satisfies the properties mentioned in lemma 3; by “ $M$  satisfies ZFC,” we mean that  $M$  satisfies these and also the axiom of choice. When  $M$  and  $N$  are transitive proper class models of ZF, then by the phrase “ $j : M \rightarrow N$  is an elementary embedding,” we mean that  $j$  is a  $\Delta_0$ -elementary cofinal embedding, which can be expressed by a single NGBC assertion. The embedding  $j$  is *trivial* if  $M = N$  and  $j$  is the identity map, and otherwise it is *nontrivial*. By these conventions, the Kunen inconsistency is expressed by a single assertion of NGBC set theory. By similar means, it can be equivalently formulated in ZFC( $j$ ) set theory as the assertion that  $j$  is not a nontrivial elementary embedding from  $V$  to  $V$ .

We conclude this section by recalling the existence of critical points for the embeddings we shall subsequently consider.

**Lemma 4.** *Suppose that  $j : M \rightarrow N$  is a nontrivial elementary embedding of transitive class models  $M$  and  $N$  satisfying ZF and having the same ordinals. If either*

- (1)  $M \models \text{AC}$ , or
- (2)  $N \subseteq M$ , or
- (3)  $M \subseteq N$  and  $M$  is definable in  $N$  without parameters or with parameters in the range of  $j$ ,

*then there is a least ordinal  $\kappa$  such that  $\kappa < j(\kappa)$ . Furthermore,  $\kappa$  is a regular uncountable cardinal in  $M$ .*

*Proof.* Statements (1) and (2) are handled in the standard set theory texts, such as [Kan04, p. 45], where we refer the reader for details. For statement (3), note first that this claim is technically a scheme, making the assertion separately of every possible definition of  $M$  in  $N$ . To see that it is true, suppose that  $M$  is a transitive class defined in  $N$  by  $x \in M \iff \varphi(x, j(b))$ , and that  $j : M \rightarrow N$  is elementary and  $j(\alpha) = \alpha$  for every ordinal  $\alpha$ . We shall argue that  $j$  is trivial, meaning  $M = N$  and  $j$  is the identity. For this, we prove that  $u \in M$



and  $j(u) = u$  by induction on the  $\in$ -rank of  $u \in N$ . Suppose that this is true for all elements of  $N$  of lower rank than a set  $u \in N$ , having rank  $\alpha$ . If  $u \in M$ , then by our induction assumption we have  $j(v) = v$  for all  $v \in u$ , and so  $u \subseteq j(u)$ . Conversely, if  $w \in j(u)$ , then since  $j(\alpha) = \alpha$ , it follows that  $w$  has rank less than  $\alpha$ , and so by induction we conclude that  $w \in M$  and  $j(w) = w$ , from which it follows that  $w \in u$ . So  $j(u) = u$ , as desired. It remains to see that every  $u$  in  $N$  of rank  $\alpha$  is in  $M$ . If not, then  $N$  thinks there is some  $w$  of rank  $\alpha$  with  $\neg\varphi(w, j(b))$ . Since  $j$  fixes  $\alpha$ , this implies by elementarity that  $M$  thinks there is  $u$  of rank  $\alpha$  with  $\neg\varphi(u, b)$ . We have already proved that  $j(u) = u$  for all such  $u$  in  $M$ . Thus, by elementarity again,  $N$  satisfies  $\neg\varphi(u, j(b))$ , contrary to the fact that  $u \in M$ . So we have proved that  $M = N$  and  $j$  is the identity, as desired.  $\square$

We note that it is not universally true that nontrivial embeddings of transitive ZF models must have critical points. Indeed, it is perfectly possible to have a nontrivial elementary embedding  $j : M \rightarrow N$  of transitive class models of ZF with no critical point. Andrés Caicedo [Cai03], for example, proves that if  $V[G]$  is the forcing extension obtained by adding  $\omega_1$  many Cohen reals, and  $G_0$  is the filter obtained from an uncountable subcollection of them, then there is an elementary embedding  $j : L(\mathbb{R}^{V[G_0]}) \rightarrow L(\mathbb{R}^{V[G]})$  which is the identity on the ordinals (and a complete argument appears on mathoverflow at [Cai]). Neeman and Zapletal [NZ98], [NZ01] showed that if there is a weakly compact Woodin cardinal, then in every small proper forcing extension  $V[G]$ , there is an elementary embedding  $j : L(\mathbb{R}^V) \rightarrow L(\mathbb{R}^{V[G]})$  that fixes every ordinal, and therefore has no critical point, but if reals are added, then since  $j(\mathbb{R}^V) = \mathbb{R}^{V[G]}$ , the embedding must be nontrivial.

## 2. ELEMENTARY EMBEDDINGS BETWEEN $V$ AND $V[G]$

Let us now begin to generalize the Kunen inconsistency by considering the possibility that nontrivial elementary embeddings  $j$  might be added by forcing. For example, perhaps one might think that in special circumstances, there could be a forcing extension  $V[G]$  in which we could find a nontrivial elementary embedding  $j : V[G] \rightarrow V$  or conversely  $j : V \rightarrow V[G]$ . The case of an embedding  $j : V[G] \rightarrow V$  is quite natural to consider, because from the perspective of the forcing extension  $V[G]$ , this would be an embedding of the universe into a certain transitive class, a situation that naïvely resembles the typical large cardinal situation. And the question of whether there can be embeddings  $j : V \rightarrow V[G]$  has arisen independently several times in the set-theoretic community. Woodin ruled out embeddings of the

form  $j : V[G] \rightarrow V$ , as we shall presently explain in theorem 5 and corollary 6. A generalization of his method, in theorem 7, rules out the converse sort of embedding  $j : V \rightarrow V[G]$ . These results directly generalize the Kunen inconsistency, which is simply the case of trivial forcing  $V[G] = V$ . Furthermore, they are themselves generalized by and special cases of theorem 8, asserting that there is no nontrivial elementary embedding between two set-forcing ground models of the universe, a result that will itself be generalized in subsequent sections.

Although logically we could skip to the most general results, we find the historical progression informative and also prefer to outline the basic methods in their easiest cases, as the generalizations eventually become complex. Nevertheless, we shall economize by giving only brief accounts in the parts of later arguments that follow a method we will have had explained earlier in detail.

In the following, let  $\text{Cof}_\delta$  be the class of ordinals having cofinality  $\delta$  and  $\text{Cof}_\delta \gamma = \gamma \cap \text{Cof}_\delta$  be the set of such ordinals below  $\gamma$ . We shall make extensive use of the Ulam-Solovay stationary partition theorem, asserting that every stationary subset of a regular uncountable cardinal  $\kappa$  can be partitioned into  $\kappa$  many stationary subsets (see [Jec03, p. 95]), a result proved for successor cardinals by Ulam and extended to all cardinals by Solovay. Note that a mild generalization of the theorem shows that if  $\omega < \kappa = \text{cof}(\lambda)$ , then every stationary subset of  $\lambda$  can be partitioned into  $\kappa$  many disjoint stationary sets. To see this, pick a normal  $\kappa$ -sequence,  $F$ , in  $\lambda$ . Then  $C \subseteq \kappa$  is a club in  $\kappa$  if and only if  $F \restriction C$  is a club in  $\lambda$ , and  $S$  is stationary in  $\kappa$  if and only if  $F \restriction S$  is stationary in  $\lambda$ . The generalization follows easily.

**Theorem 5** (Woodin). *In any set-forcing extension  $V[G]$ , there is no nontrivial elementary embedding  $j : V[G] \rightarrow V$ .*

*Proof.* Suppose towards contradiction that  $V[G]$  is a set-forcing extension of  $V$ , obtained by forcing with  $\mathbb{P} \in V$  to add the  $V$ -generic filter  $G \subseteq \mathbb{P}$ , and that  $j : V[G] \rightarrow V$  is a nontrivial elementary embedding in  $V[G]$ , where  $j$  is a class in  $V[G]$ . By lemma 4 applied in  $V[G]$ , it follows that  $j$  has a critical point,  $\kappa$ , which of course will be a measurable cardinal in  $V[G]$ .

First, we find an ordinal  $\lambda$  above both  $\kappa$  and  $|\mathbb{P}|$  such that  $j(\lambda) = \lambda$  and  $\text{cof}(\lambda)^V = \text{cof}(\lambda)^{V[G]} = \omega$ . To find  $\lambda$ , fix any ordinal  $\delta_0$  above  $\kappa$  and  $|\mathbb{P}|$  with  $\text{cof}(\delta_0) = \omega$  in  $V$ . If  $j(\delta_0) = \delta_0$ , then we have found what we wanted. Otherwise,  $\delta_0 < j(\delta_0)$  and we may recursively define  $\delta_{n+1} = j(\delta_n)$  and  $\lambda = \sup\{\delta_n \mid n \in \omega\}$ . Since  $\lambda$  is the supremum of the increasing  $\omega$ -sequence  $\langle \delta_n \mid n < \omega \rangle$ , it follows that  $j(\lambda)$  is the supremum

of  $j(\langle \delta_n \mid n < \omega \rangle) = \langle j(\delta_n) \mid n < \omega \rangle = \langle \delta_{n+1} \mid n < \omega \rangle$ . Thus,  $j(\lambda) = \lambda$  and  $\text{cof}(\lambda) = \omega$  in both  $V[G]$  and  $V$ , as desired.

Since  $\lambda > |\mathbb{P}|$ , it follows that  $\mathbb{P}$  satisfies the  $\lambda$ -c.c. in  $V$  and so  $\lambda^+$  has the same meaning in both  $V$  and  $V[G]$ . Thus,  $j(\lambda^+) = \lambda^+$ . The set  $\text{Cof}_\omega \lambda^+$  of  $V[G]$  is stationary in  $V[G]$ , and so by the Ulam-Solovay Theorem, we may partition it into  $\kappa$  many disjoint stationary sets:  $\text{Cof}_\omega \lambda^+ = \bigsqcup_{\alpha < \kappa} S_\alpha$  in  $V[G]$ . Let  $\vec{S} = \langle S_\alpha \mid \alpha < \kappa \rangle$  and consider  $S^* = j(\vec{S})(\kappa)$ . This is a stationary subset of  $(\text{Cof}_\omega \lambda^+)^V$  in  $V$ , and it is disjoint from  $j(S_\alpha)$  for all  $\alpha < \kappa$ . Let  $C = \{ \beta < \lambda^+ \mid j \restriction \beta \subseteq \beta \}$ , which is a club subset of  $\lambda^+$  in  $V[G]$ . Since  $\lambda^+$  is above the size of the forcing, it follows that there is a club subset  $D \subseteq C$  with  $D \in V$ . Thus, there is some  $\beta \in D \cap S^*$ . Since  $S^* \subseteq \text{Cof}_\omega \lambda^+$ , it follows that  $\text{cof}(\beta) = \omega$  in  $V$  and hence also in  $V[G]$ . Since  $j \restriction \beta \subseteq \beta$  and  $\text{cof}(\beta) = \omega$ , it follows easily that  $j(\beta) = \beta$ . But since  $\beta \in (\text{Cof}_\omega \lambda^+)^{V[G]}$ , there must be some  $\alpha < \kappa$  with  $\beta \in S_\alpha$ , since these sets form a partition. Thus,  $\beta = j(\beta) \in j(S_\alpha) = j(\vec{S})(j(\alpha))$  and  $\beta \in S^* = j(\vec{S})(\kappa)$ , contrary to the fact that the sets appearing in  $j(\vec{S})$  are disjoint. So there can be no such embedding  $j$ , and the proof is complete.  $\square$

An equivalent formulation of theorem 5, stated from the point of view of the extension, is the following.

**Corollary 6.** *If  $j : V \rightarrow M$  is a nontrivial elementary embedding in  $V$ , then  $V$  is not a set-forcing extension of  $M$ .*

In other words, if  $j : V \rightarrow M$ , then  $M$  is not a set-forcing ground model of  $V$ .

Let us turn now to the converse sort of embedding, from  $V$  to  $V[G]$ , which we shall rule out by a generalization of the method. In the case where  $V[G]$  is the target of the embedding rather than its domain, it no longer suffices to consider  $\text{Cof}_\omega \lambda^+$  in the domain of the embedding, because an ordinal of cofinality  $\omega$  in  $V[G]$  might have a higher cofinality in  $V$ , and the argument breaks down when applied directly. We shall repair this issue by considering a stationary set of ordinals having much larger cofinality. Another subtle point is that the previous argument to obtain the fixed point  $\lambda$  no longer succeeds exactly as before in the new context, because the sequence  $\langle \delta_n \mid n \in \omega \rangle$  may not be an element of  $V$ ; but again a repair will be provided by considering a longer sequence.

Attribution for this next theorem is not clear to us. Woodin reportedly proved it along with theorem 5 while he was a graduate student in the early 1980s. But also, Matt Foreman mentioned to the first author that he discussed a version of the theorem with Mack Stanley and Sy Friedman around the same time, but their proof was evidently

different than ours, and unfortunately the result was not published.<sup>2</sup> Suzuki proved a theorem implying our theorem 7 in [Suz98, p. 344], using a technique essentially the same as ours.<sup>3</sup> The question about such embeddings has arisen several times since then, however, and so we are pleased to provide a proof here.

**Theorem 7.** *In any set-forcing extension  $V[G]$ , there is no nontrivial elementary embedding  $j : V \rightarrow V[G]$ .*

*Proof.* Suppose towards contradiction that  $V[G]$  is a set-forcing extension, obtained by forcing with  $\mathbb{P}$  to add a  $V$ -generic filter  $G \subseteq \mathbb{P}$ , and that  $j : V \rightarrow V[G]$  is an elementary embedding and a class in  $V[G]$ . Since  $V \models \text{ZFC}$ , it follows by lemma 4 that  $j$  has a critical point  $\kappa$ , which it is easy to see must be a regular uncountable cardinal in  $V$ .

We claim as in theorem 5 that there is an ordinal  $\lambda$  above  $\kappa$  and  $|\mathbb{P}|$  with  $\text{cof}(\lambda)^V = \omega$  and  $j(\lambda) = \lambda$ . To see this, consider the class function  $j \upharpoonright \text{ORD} : \text{ORD} \rightarrow \text{ORD}$  in  $V[G]$ . As before, start at any  $\delta_0$  greater than both  $\kappa$  and  $|\mathbb{P}|$  and with  $\text{cof}(\delta_0)^V = \omega$ . If  $j(\delta_0) = \delta_0$ , we are done. Otherwise, define  $\delta_{\alpha+1} = j(\delta_\alpha + 1)$  and  $\delta_\xi = \sup_{\alpha < \xi} \delta_\alpha$  for limit  $\xi$ . Because we added 1 at each step, this is a strictly increasing, continuous sequence of ordinals, and by construction,  $j \restriction \delta_\xi \subseteq \delta_\xi$  for any limit  $\xi$ . Let  $\gamma = |\mathbb{P}|^+$  and  $C = \{\delta_\xi \mid \xi < \gamma\}$ . This is a club subset of  $\delta_\gamma$ , which has cofinality  $\gamma$ . Since  $\mathbb{P}$  satisfies the  $\gamma$ -c.c., it follows that there is a club  $D \subseteq C$  with  $D \in V$ . Let  $\lambda$  be the  $\omega^{\text{th}}$  element of  $D$ . Thus,  $\text{cof}(\lambda)^V = \omega$  and since  $\lambda \in C$ , we know that  $j \restriction \lambda \subseteq \lambda$ . Since  $\lambda$  is the supremum of an  $\omega$ -sequence in  $V$ , it follows that  $j(\lambda)$  is the supremum of  $j$  of that sequence, but since  $j \restriction \lambda \subseteq \lambda$ , this supremum is just  $\lambda$ . So  $j(\lambda) = \lambda$ , as desired.

Since  $j(\lambda) = \lambda$  and all cardinals above  $\lambda$  are preserved by the forcing, it follows that  $j(\lambda^+) = \lambda^+$  and  $j(\lambda^{++}) = \lambda^{++}$ . Since  $\lambda^+$  is a regular cardinal larger than  $|\mathbb{P}|$ , it follows that  $\text{Cof}_{\lambda^+}$  is absolute between  $V$  and  $V[G]$ . Since  $\text{Cof}_{\lambda^+} \lambda^{++}$  is stationary, it follows by the Ulam-Solovay theorem in  $V$  that we may partition it as a disjoint union:  $\text{Cof}_{\lambda^+} \lambda^{++} = \bigsqcup_{\alpha < \kappa} S_\alpha$  of  $\kappa$  many stationary subsets  $S_\alpha$ . Let  $\vec{S} = \langle S_\alpha \mid \alpha < \kappa \rangle$  and

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<sup>2</sup>Part of their focus was reportedly on the extent to which the result generalized to class forcing. For example, they considered the case of class forcing extensions by amenable class forcing. Foreman mentioned that Woodin has an example of forcing using a class version of non-stationary tower forcing where  $j : V \rightarrow V[G]$ , but  $V[G]$  does not have ZFC for the predicate  $V$ , a result cited in [Suz99, p.1594] and in [VW01, p. 1091].

<sup>3</sup>Suzuki proved that if there is  $j : V \rightarrow M$  in  $V[G]$ , then  $V \not\subseteq M$ . This result is stronger than our theorem 7, but weaker than our theorem 10. Although Suzuki states in his introduction that his proof only concerns definable  $j$ , in fact his proof never uses that fact and can be formalized in NGBC.

consider  $S^* = j(\vec{S})(\kappa)$ . By elementarity, this is a stationary subset of  $\text{Cof}_{\lambda^+} \lambda^{++}$  in  $V[G]$ , and disjoint from  $j(S_\alpha)$  for all  $\alpha < \kappa$ . Let  $C = \{\beta < \lambda^{++} \mid j \restriction \beta \subseteq \beta\}$ , which is a club subset of  $\lambda^{++}$  in  $V[G]$ . Thus, there is some  $\beta \in S^* \cap C$ . In particular,  $\beta \in S^* \subseteq \text{Cof}_{\lambda^+}$  and so  $\text{cof}(\beta) = \lambda^+$  in  $V[G]$  and in  $V$ . If  $\beta$  is the supremum of  $s \subseteq \beta$ , where  $s$  has order type  $\lambda^+$ , it follows that  $j(\beta)$  is the supremum of  $j(s)$ , having order type  $j(\lambda^+) = \lambda^+$ . It follows that  $j \restriction s$  is unbounded in  $j(s)$ , and therefore since  $j \restriction \beta \subseteq \beta$ , it follows that the supremum of  $j(s)$  is  $\beta$ . Thus,  $j(\beta) = \beta$ . Since  $\beta \in \text{Cof}_{\lambda^+} \lambda^{++}$ , it must be that  $\beta \in S_\alpha$  for some  $\alpha < \kappa$ . So we have a contradiction, just as in the proof of theorem 5, since  $\beta$  is in both  $j(\vec{S})(\alpha)$  and in  $j(\vec{S})(\kappa)$ , even though these are disjoint.  $\square$

Stated from the perspective of the forcing extension, what theorem 7 asserts is that if  $j : M \rightarrow V$  is a nontrivial elementary embedding from a transitive class  $M$  into the universe  $V$ , then  $V$  is not a set-forcing extension of  $M$ . Both theorem 5 and 7 are therefore special cases of the following theorem.

**Theorem 8.** *If  $M$  and  $N$  are set-forcing ground models of  $V$ , then there is no nontrivial elementary embedding  $j : M \rightarrow N$ .*

In other words, if  $M$  and  $N$  have a common set-forcing extension  $M[G] = N[H]$ , then in no such extension is there a nontrivial elementary embedding  $j : M \rightarrow N$ . This theorem is an immediate corollary of theorem 10 in the next section, since any ground model is stationary correct in its forcing extension at regular cardinals above the size of the forcing. An important special case of theorem 8, obtained simply by applying it in a forcing extension  $V[G]$ , is the generic embedding version of the Kunen inconsistency.

**Corollary 9.** *In no set-forcing extension  $V[G]$  is there a nontrivial elementary embedding  $j : V \rightarrow V$ .*

Indeed, theorem 10 will show that there is no such class  $j$  in any extension of  $V$  that is eventually stationary correct (this includes many class forcing extensions), and the remarks after that theorem generalize it still further.

### 3. ELEMENTARY EMBEDDINGS BETWEEN EVENTUALLY STATIONARY-CORRECT MODELS

We define that a transitive class model  $M$  of set theory is *stationary correct* at a limit ordinal  $\delta$  of  $M$  if every subset of  $\delta$  that is stationary in  $M$  remains stationary in the universe  $V$ . More generally, one model  $M$

is stationary correct to another larger model  $N$  at  $\delta$ , if every stationary subset of  $\delta$  in  $M$  is stationary in  $N$ . For example, after any forcing of size less than a regular cardinal  $\delta$ , and indeed, after any  $\delta$ -c.c. forcing, every stationary subset of  $\delta$  in the ground model remains stationary in the extension, because every club subset of  $\delta$  in the forcing extension will contain a club of the ground model as a subset. After any set forcing, therefore, the ground model is stationary correct in the extension at all regular cardinals above the size of the forcing. Since in general regular cardinals of  $M$  can become singular in  $V$ , let us adopt the convention that a subset  $S \subseteq \delta$  is stationary in the case  $\text{cof}(\delta) = \omega$  if and only if  $S$  contains a final segment  $(\alpha, \delta)$  for some  $\alpha < \delta$ . In particular, in the extreme case that a cardinal  $\delta$  has uncountable cofinality in  $M$  and countable cofinality in  $V$ , then  $M$  is not stationary correct at  $\delta$ .

Before proving the theorem, we make a few elementary observations about stationary correctness. First, a transitive class  $M$  is stationary correct at  $\delta$  if and only if it is stationary correct at  $\text{cof}(\delta)^M$ . The countable cofinality case is immediate; when the cofinality is uncountable in  $M$ , then in either case of stationary-correctness, it follows that  $\delta$  also has uncountable cofinality in  $V$ . We may fix a club subset  $C \subseteq \delta$  in  $M$  with order type  $\text{cof}(\delta)^M$  and observe that a set  $S \subseteq \delta$  is stationary if and only if  $S \cap C$  is stationary. Viewing  $S \cap C$  as a subset of  $C$ , this latter claim is equivalent to the stationarity of a subset of  $\text{cof}(\delta)^M$ , as desired. Second, we note that if  $M \subseteq V$  is stationary correct at a regular cardinal  $\delta$  of  $M$ , then  $\delta$  remains regular in  $V$ . If not, let  $\eta = \text{cof}(\delta)^V < \delta$  be the new smaller cofinality, which is a regular cardinal in  $V$ , and let  $C \subseteq \delta$  be a cofinal  $\eta$ -sequence in  $V$ . The case  $\eta = \omega$  is easily ruled out by our convention about stationary subsets of ordinals with cofinality  $\omega$ , and so we may assume  $\eta$  is uncountable. The set  $C'$  of limit points of  $C$ , therefore, is club and consists entirely of ordinals of cofinality less than  $\eta$  in  $V$ . Since  $\eta$  is regular in  $V$ , no such ordinal can have cofinality  $\eta$  in  $M$ , and so  $C'$  is disjoint from the set  $(\text{Cof}_\eta \delta)^M$ . This set, which is stationary in  $M$ , is thus no longer stationary in  $V$ , contradicting our hypothesis that  $M$  was stationary correct. Finally, third, the two previous observations together imply that if  $M \subseteq V$  is stationary correct at an ordinal  $\delta$ , then  $\text{cof}(\delta)^M = \text{cof}(\delta)^V$ .

**Theorem 10.** *Suppose that  $M$  and  $N$  are transitive class models of ZFC and both are stationary-correct at all sufficiently large regular cardinals of  $M$  and  $N$ . Then in  $V$  there is no nontrivial elementary embedding  $j : M \rightarrow N$ .*

*Proof.* Suppose that  $j : M \rightarrow N$  is a nontrivial elementary embedding and that  $M$  and  $N$  are transitive proper class models of ZFC, which are

eventually stationary correct to  $V$ . By lemma 4, the embedding  $j$  has a critical point  $\kappa$ . As in the previous proofs, we shall begin by finding an ordinal  $\lambda$  above the critical point of  $j$  with  $j(\lambda) = \lambda$ . To find such an ordinal, suppose that  $M$  and  $N$  are stationary correct at all regular cardinals of  $M$  or  $N$  above  $\theta$ , and we may assume  $\kappa < \theta$ . The general considerations before the theorem show that if a cardinal should have cofinality above  $\theta$  in any of the models  $M$ ,  $N$  or  $V$ , then this cofinality is preserved to the other models. Considering the operation of  $j$  on the ordinals, it is easy to see that  $C = \{\beta \in \text{ORD} \mid j \restriction \beta \subseteq \beta\}$  is a closed unbounded proper class of ordinals. Thus, there is some  $\eta \in C$  with  $\theta < \text{cof}(\eta)$ . Since  $(\text{Cof}_\omega \eta)^M$  is stationary in  $M$ , it remains stationary in  $V$ . Since  $C \cap \eta$  is club in  $\eta$ , there is some  $\lambda \in C \cap (\text{Cof}_\omega \eta)^M$  with  $\theta \leq \lambda$ . Thus,  $j \restriction \lambda \subseteq \lambda$  and  $\text{cof}(\lambda)^M = \omega$ . From this, it easily follows that  $j(\lambda) = \lambda$ . Since cardinals above  $\theta$  are preserved, it follows that  $\lambda^+$  and  $\lambda^{++}$  are the same in all three models, and so  $j(\lambda^+) = \lambda^+$  and  $j(\lambda^{++}) = \lambda^{++}$ .

We continue the argument just as in theorem 7. Since cofinalities above  $\theta$  are computed correctly, the set  $\text{Cof}_{\lambda^+} \lambda^{++}$  is absolute between  $M$ ,  $N$  and  $V$ . By the Ulam-Solovay Theorem, there is a partition  $\text{Cof}_{\lambda^+} \lambda^{++} = \bigsqcup_{\alpha < \kappa} S_\alpha$  in  $M$  into stationary sets  $S_\alpha$ . Let  $\vec{S} = \langle S_\alpha \mid \alpha < \kappa \rangle$  and  $S^* = j(\vec{S})(\kappa)$ . By elementarity,  $S^*$  is a stationary subset of  $\text{Cof}_{\lambda^+} \lambda^{++}$  in  $N$ , and hence also stationary in  $V$ . Since  $C \cap \lambda^{++}$  is club in  $\lambda^{++}$ , there is some  $\beta \in C \cap S^*$ . Thus,  $j \restriction \beta \subseteq \beta$  and  $\text{cof}(\beta) = \lambda^+$  in  $N$ . Since this is above  $\theta$ , it follows that  $\text{cof}(\beta) = \lambda^+$  also in  $V$  and  $M$ , and from this it follows as before that  $j(\beta) = \beta$ . Since  $\beta$  has cofinality  $\lambda^+$  in  $M$ , it follows that  $\beta \in S_\alpha$  for some  $\alpha < \kappa$ , which implies that  $\beta \in j(S_\alpha)$  and in  $S^*$ , contradicting the fact that these sets are disjoint.  $\square$

Because every ground model is stationary correct in its set-forcing extensions above the size of the forcing, it follows that theorem 8 is a special case of theorem 10, and therefore theorem 10 generalizes all the theorems of section 2. This theorem, however, also allows us to rule out embeddings between certain class forcing ground models, provided that they are eventually stationary correct, and this includes many natural class-forcing iterations.

It is clear that we may weaken the assumption that  $M$  and  $N$  are eventually stationary correct in theorem 10, since once we knew that the models agreed on  $\lambda^+$  and  $\lambda^{++}$ , then stationarity was used only to ensure that the sets had a member in the particular club class  $C$ , the class of closure points of  $j$ .

For example, for a class function  $j$ , we could define that a class  $M$  is  $j$ -correct, if it is eventually correct about regular cardinals and there is a closed unbounded class  $D$  of ordinals such that if  $C = \{\beta \in D \mid j \restriction \beta \subseteq \beta\}$  is unbounded in some  $\gamma$ , then every  $S \subseteq \gamma$  that  $M$  thinks is stationary contains an element of  $C$ . The concept of stationary correctness in theorem 10 and many of the other theorems in this article could then be replaced with the concept of  $j$ -correctness. For example, theorem 10 would become the claim that there is no non-trivial elementary embedding  $j : M \rightarrow N$ , for which  $M$  and  $N$  are both  $j$ -correct.

#### 4. EMBEDDINGS FROM $V$ INTO HOD AND RELATED MODELS

Let us consider now the possibility of a nontrivial elementary embedding  $j : V \rightarrow \text{HOD}$  of the universe into the class of hereditarily ordinal-definable sets. After ruling out such embeddings in theorem 11, we shall generalize the result to several other related inner models. We shall give a slightly modified version of Woodin's original proof of this theorem, which will enable us easily to extract additional information from it in the various corollaries and theorems that we prove after it in this section.

**Theorem 11** (Woodin). *There is no nontrivial elementary embedding  $j : V \rightarrow \text{HOD}$ .*

*Proof.* We note briefly that we need not specifically assume AC in this argument, as it follows trivially that  $V$  satisfies AC from the assumption that  $j : V \rightarrow \text{HOD}$  is elementary, since AC holds in HOD. By lemma 4, the embedding  $j$  has a critical point  $\kappa$ , which must be a measurable cardinal in  $V$ . Define the usual critical sequence by  $\kappa_0 = \kappa$  and  $\kappa_{n+1} = j(\kappa_n)$ , and let  $\lambda = \sup_{n < \omega} \kappa_n$ . As in the previous proofs, it follows that  $j(\lambda) = \lambda$  and also that  $j(\lambda^+) = \lambda^+$ . By the Ulam-Solovay Theorem, we may partition  $\text{Cof}_\omega \lambda^+$  into  $\lambda^+$  many disjoint stationary sets  $\vec{S} = \langle S_\alpha \mid \alpha < \lambda^+ \rangle$ , with  $\text{Cof}_\omega \lambda^+ = \bigsqcup_{\alpha < \lambda^+} S_\alpha$ . Let  $\vec{T} = j(\vec{S}) = \langle T_\alpha \mid \alpha < \lambda^+ \rangle$ .

We claim for  $\xi < \lambda^+$  that  $\xi \in \text{ran}(j)$  if and only if  $T_\xi$  is stationary in  $V$ . For the forward direction, suppose that  $\xi = j(\alpha)$ . Let  $C = \{\beta < \lambda^+ \mid j \restriction \beta \subseteq \beta\}$ , which is a club subset of  $\lambda^+$  in  $V$ . Observe that if  $\beta \in C$  and  $\text{cof}(\beta) = \omega$ , then  $j(\beta) = \beta$ . Thus, if  $\beta \in C \cap \text{Cof}_\omega \lambda^+$ , then  $\beta \in S_\alpha \iff \beta = j(\beta) \in j(S_\alpha) = T_{j(\alpha)}$ . Since  $S_\alpha$  and  $T_{j(\alpha)}$  are contained within the ordinals of cofinality  $\omega$ , this means that  $C \cap S_\alpha = C \cap T_{j(\alpha)}$ . In short,  $S_\alpha$  and  $T_{j(\alpha)}$  agree on a club, and so  $T_{j(\alpha)}$  is stationary, as desired. Conversely, suppose that  $T_\xi$  is



stationary in  $V$ . It follows that there is some  $\beta \in C \cap T_\xi$ . Since every element of  $T_\xi$  has cofinality  $\omega$ , it follows that  $j(\beta) = \beta$ . But since  $\beta \in \text{Cof}_\omega \lambda^+$ , we must also have  $\beta \in S_\alpha$  for some  $\alpha < \lambda^+$ . It follows that  $\beta = j(\beta) \in j(S_\alpha) = T_{j(\alpha)}$ . So  $T_\xi$  and  $T_{j(\alpha)}$  have the element  $\beta$  in common, and since  $\vec{T}$  is a partition, it must be that  $\xi = j(\alpha)$ .

The claim of the previous paragraph shows that  $j \restriction \lambda^+$  is definable from  $\vec{T}$ , which is in  $\text{HOD}$ , and so  $j \restriction \lambda^+ \in \text{HOD}$  as well. From  $j \restriction \lambda^+$ , we can also define  $j \restriction \lambda^+$ , and so  $C \in \text{HOD}$ . To complete the argument, let  $S^* = T_\kappa$ , which is a subset of  $\text{Cof}_\omega \lambda^+$  that is stationary in  $\text{HOD}$ . Since  $C$  is a club in  $\text{HOD}$ , there is  $\beta \in C \cap S^*$ . Since  $\text{cof}(\beta) = \omega$  and  $\beta \in C$ , it follows that  $j(\beta) = \beta$ . Since  $\beta \in \text{Cof}_\omega \lambda^+$ , it follows that  $\beta \in S_\alpha$  for some  $\alpha < \lambda^+$ . Thus,  $\beta = j(\beta) \in j(S_\alpha) = T_{j(\alpha)}$ , contradicting the fact that  $T_\kappa$  and  $T_{j(\alpha)}$  are disjoint.  $\square$

Note that once we established  $j \restriction \lambda^+ \in \text{HOD}$ , we could have finished the proof instead by using Kunen's original method with  $\omega$ -Jónsson functions or by using the technique of Harada [Kan04, p. 320-321] or of Zapletal [Zap96].

For any class  $A$ , we define  $\text{HOD}(A)$  to be the class of sets that are definable using ordinal parameters and parameters in  $A$ . This differs somewhat from the more generous class defined in [Jec03, p.195], where it is also allowed that the class  $A$  appears in the definitions as a predicate, but as we do not need this feature, our results are stronger with the more restrictive notion.

**Theorem 12.** *If  $j : V \rightarrow M$  is an elementary embedding of  $V$  into an inner model  $M$ , then  $V = \text{HOD}(M)$ . That is, every object in  $V$  is definable in  $V$  using a parameter from  $M$ . Furthermore, every object in  $V$  is definable using a parameter from the image of  $j$ .*

*Proof.* This theorem can be viewed as a corollary to the proof of theorem 11 (although one could reverse this and view theorem 11 as a corollary to this result, arguing that if  $j : V \rightarrow \text{HOD}$ , then  $V = \text{HOD}(\text{HOD})$ , which means  $V = \text{HOD}$ , which prevents such a  $j$ ). Suppose  $j : V \rightarrow M$  is elementary. We shall first prove that for every ordinal  $\gamma$ , the restriction  $j \restriction \gamma$  is definable in  $V$  from parameters in  $M$ . As in the proof of theorem 11, we may find  $\lambda > \gamma$  with  $j(\lambda) = \lambda$  and  $\text{cof}(\lambda) = \omega$ , by iterating the embedding  $\omega$  many times above  $\gamma$ . It follows that  $j(\lambda^+) = \lambda^+$ . And as in that proof, we may again partition  $\text{Cof}_\omega \lambda^+ = \bigsqcup_{\alpha < \lambda^+} S_\alpha$  into stationary sets  $\vec{S} = \langle S_\alpha \mid \alpha < \lambda^+ \rangle$ , and, setting  $\vec{T} = j(\vec{S}) = \langle T_\alpha \mid \alpha < \lambda^+ \rangle$ , observe for  $\xi < \lambda^+$  that  $\xi \in \text{ran}(j)$  if and only if  $T_\xi$  is stationary in  $V$ . Thus,  $j \restriction \lambda^+$  and hence  $j \restriction \gamma$  is definable in  $V$  from  $\vec{T}$ , which is an element of  $M$ . In particular,  $j \restriction \gamma$  is

definable from  $\vec{T}$  and  $\gamma$ . This proof also shows that for an unbounded class of successor cardinals  $\lambda^+$ ,  $j \upharpoonright \lambda^+$  is definable from parameters in the image of  $j$ , since  $\vec{T}$  is in the range of  $j$ .

Since any set  $A \subseteq \lambda^+$  is easily definable from  $j(A)$  and  $j \restriction \lambda^+$ , it follows that every set of ordinals in  $V$  is definable in  $V$  from parameters in the range of  $j$ . Since every set is coded by a set of ordinals, it follows that every set in  $V$  is definable in  $V$  from parameters in the range of  $j$ , and in particular,  $V = \text{HOD}(M)$ . The proof is thus complete.  $\square$

Theorem 12 shows in particular for any set  $A$  that  $j \upharpoonright A$  is definable in  $V$  using parameters from  $M$ .

**Corollary 13.** *If  $j : V \rightarrow M$  is an elementary embedding of  $V$  into a transitive class  $M$  for which  $M \subseteq \text{HOD}$ , then  $V = \text{HOD}$ .*

*Proof.* Note that we need not assume AC for this result, since our hypothesis actually implies it: defining that  $r$  precedes  $s$  if and only if  $j(r)$  precedes  $j(s)$  in the HOD order pulls back the definable HOD order from  $M$  to a global well-ordering of  $V$ . The claim is now immediate from theorem 12, since if  $M \subseteq \text{HOD}$ , then  $\text{HOD}(M) = \text{HOD}$ , since having parameters from  $M$  doesn't help beyond having ordinal parameters.  $\square$

In other words, corollary 13 asserts that if  $V \neq \text{HOD}$  and  $M \subseteq \text{HOD}$ , then there is no nontrivial  $j : V \rightarrow M$ .

Using the ideas of theorem 7, we shall now improve theorem 12 to the case of embeddings  $j : M \rightarrow N$ , not necessarily defined on all of  $V$ , provided that  $M$  is eventually stationary correct. This will allow us afterward to generalize the previous corollaries to the case of the generic HOD.

**Theorem 14.** *Suppose that  $j : M \rightarrow N$  is a nontrivial elementary embedding between inner models  $M$  and  $N$  of ZFC and that  $M$  is eventually stationary correct to  $V$ . Then  $M \subseteq \text{HOD}(N)$  and furthermore  $j \upharpoonright A \in \text{HOD}(N)$  for any  $A \in M$ . Indeed, for every  $A \in M$ ,  $A$  and  $j \upharpoonright A$  are definable in  $V$  from parameters in the range of  $j$ .*

*Proof.* Suppose  $j : M \rightarrow N$  is elementary, where  $M$  and  $N$  are inner models of ZFC and  $M$  is stationary correct to  $V$  above  $\theta$ . Fix any ordinal  $\gamma$ . As in the proof of theorem 10, we may find an ordinal  $\lambda$  above both  $\theta$  and  $\gamma$  and of cofinality  $\omega$  in  $M$  such that  $j(\lambda) = \lambda$  and  $\lambda > \theta$ . Since  $M$  is stationary correct above  $\theta$ , it follows that  $\lambda^+$  and  $\lambda^{++}$  are the same in  $M$  and  $V$ , and from this it follows that  $j(\lambda^+) = \lambda^+$  and  $j(\lambda^{++}) = \lambda^{++}$ , and so all three models  $M$ ,  $N$  and  $V$  agree on  $\lambda^+$ ,  $\lambda^{++}$  and the set  $\text{Cof}_{\lambda^+} \lambda^{++}$ . By the Ulam-Solovay

theorem, we may partition  $\text{Cof}_{\lambda^+} \lambda^{++} = \bigsqcup_{\alpha < \gamma} S_\alpha$  into stationary sets  $\vec{S} = \langle S_\alpha \mid \alpha < \lambda^{++} \rangle$  in  $M$ . Let  $\vec{T} = \langle T_\alpha \mid \alpha < \lambda^{++} \rangle = j(\vec{S})$ .

We claim as in theorem 12 for  $\xi < \lambda^{++}$  that  $\xi \in \text{ran}(j)$  if and only if  $T_\xi$  is stationary in  $V$ . If  $\xi = j(\alpha)$ , then as in the proof of theorem 11, we know that  $S_\alpha$  and  $T_\xi$  agree on the club  $C = \{\beta < \lambda^{++} \mid j \restriction \beta \subseteq \beta\}$ , since any such  $\beta$  of cofinality  $\lambda^+$  is fixed by  $j$ , and so  $T_\xi$  is stationary in  $V$ . Conversely, if  $T_\xi$  is stationary in  $V$ , then there is some  $\beta \in T_\xi \cap C$ , and so  $\text{cof}(\beta) = \lambda^+$  in  $N$  and hence  $V$  and  $M$ , and so  $\beta \in S_\alpha$  for some ordinal  $\alpha < \lambda^{++}$ . It follows that  $\beta = j(\beta) \in T_{j(\alpha)}$ , which implies  $\xi = j(\alpha)$  since the sets in  $\vec{T}$  are disjoint.

Thus,  $j \restriction \lambda^{++}$  and hence  $j \restriction \lambda^{++}$  is definable in  $V$  from  $\vec{T}$ , which is an element of the range of  $j$ . Since any set  $A \subseteq \lambda^{++}$  in  $M$  is easily definable from  $j(A)$  and  $j \restriction \lambda^{++}$ , it follows that every set of ordinals in  $M$  is definable in  $V$  from parameters in  $N$ , indeed from parameters in the image of  $j$ . Similarly,  $j \restriction A$  is definable from  $A$  and  $j \restriction \lambda^{++}$ . Since every set in  $M$  is coded by a set of ordinals in  $M$ , it follows that every set  $A$  in  $M$  is definable in  $V$  from parameters in the image of  $j$ , so in particular,  $M \subseteq \text{HOD}(N)$ . By enumerating the set  $A$ , one may similarly conclude that  $j \restriction A$  is definable from parameters in the range of  $j$  as desired.  $\square$

**Corollary 15.** *There is no generic embedding  $j : V \rightarrow \text{HOD}$ . That is, in no set-forcing extension  $V[G]$  is there a nontrivial elementary embedding  $j : V \rightarrow \text{HOD}^V$ .*

*Proof.* Suppose that  $j : V \rightarrow \text{HOD}^V$  is a nontrivial elementary embedding in a set-forcing extension  $V[G]$ . By performing additional collapse forcing, we may assume that the forcing is almost homogeneous and ordinal definable. By theorem 14, it follows that every element of  $V$  is definable in  $V[G]$  using parameters in  $\text{HOD}^V$ , that is,  $V \subseteq \text{HOD}(\text{HOD}^V)^{V[G]}$ . But  $\text{HOD}(p)^{V[G]} \subseteq \text{HOD}(p)^V$  for any parameter  $p \in V$ , by our assumption on the forcing, and so  $\text{HOD}(\text{HOD}^V)^{V[G]} = \text{HOD}^V$ . So we've argued that  $V \subseteq \text{HOD}^V$  and hence  $V = \text{HOD}^V$ . The nonexistence of  $j$  now follows from corollary 9.  $\square$

Let us now explain how this analysis extends to the case of the iterated HOD classes  $\text{HOD}^\eta$ , obtained by iteratively relativizing the HOD class. To define these classes, we begin with  $\text{HOD}^0 = V$  and define  $\text{HOD}^{n+1} = \text{HOD}^{\text{HOD}^n}$ , so that  $\text{HOD}^1$  is simply HOD and  $\text{HOD}^2 = \text{HOD}^{\text{HOD}}$  is HOD as computed inside HOD, and so on. We would naturally want to continue this iteration with an intersection  $\text{HOD}^\omega = \bigcap_{n < \omega} \text{HOD}^n$  at  $\omega$ , but for a subtle metamathematical reason, this may

not succeed in defining a class. The reason is that our previous definition of  $\text{HOD}^n$  was by a *meta-theoretic* induction on  $n$ ; although each  $\text{HOD}^n$  is a definable class for any meta-theoretic natural number  $n$ , these definitions become progressively more complex as  $n$  increases, and the definition does not provide a uniform presentation of the  $\text{HOD}^n$ , which we seem to need in order to take the intersection  $\bigcap_n \text{HOD}^n$ . Indeed, a 1974 result of Harrington appearing in [Zad83, section 7], with related work in [McA74], shows that it is consistent with NGBC that  $\text{HOD}^n$  exists for each  $n < \omega$  but the intersection  $\text{HOD}^\omega$  is not a class. Nevertheless, some models have a special structure allowing them to enjoy a uniform definition of these classes, and in these models we may continue the iteration transfinitely. To illuminate this situation, we define that a class  $H$  is a *uniform presentation* of  $\text{HOD}^\alpha$  for  $\alpha < \eta$  if  $H \subseteq \{(x, \alpha) \mid \alpha < \eta\}$  and the slices  $H^\alpha = \{x \mid (x, \alpha) \in H\}$  for  $\alpha < \eta$  are all models of ZF and obey the defining properties of  $\text{HOD}^\alpha$ , namely, the base case  $H^0 = V$ , the successor case  $H^{\alpha+1} = \text{HOD}^{H^\alpha}$  and the limit case  $H^\gamma = \bigcap_{\alpha < \gamma} H^\alpha$  for limit ordinals  $\gamma$ . By induction, any two such classes  $H$  agree on their common coordinates, and we shall write  $\text{HOD}^\alpha$  to mean  $H^\alpha$  for such a class  $H$ , which will be fixed in the background for a given discourse. Let us define the phrase “ $\text{HOD}^\eta$  exists” to mean that  $\eta$  is an ordinal and there is a uniform presentation of  $\text{HOD}^\alpha$  for  $\alpha \leq \eta$ . This is nearly equivalent to the existence of a uniform presentation of  $\text{HOD}^\alpha$  for  $\alpha < \eta$ , since the missing class  $\text{HOD}^\eta$  can be computed from that information: if  $\eta$  is a limit ordinal, then  $\text{HOD}^\eta$  is the intersection of all  $\text{HOD}^\alpha$  for  $\alpha < \eta$ , and if this is a ZF model, then we can also say that  $\text{HOD}^\eta$  exists; and if  $\eta = \beta + 1$  is a successor ordinal, then  $\text{HOD}^\eta = \text{HOD}^{\text{HOD}^\beta}$ , so  $\text{HOD}^\eta$  exists. It is easy to see that  $\text{HOD}^n$  exists for any (meta-theoretic) natural number  $n$ , and if  $\text{HOD}^\eta$  exists, so does  $\text{HOD}^{\eta+1}$  and  $\text{HOD}^\alpha$  for any  $\alpha < \eta$ . But by the Harrington result we mentioned earlier, one cannot necessarily proceed through limit ordinals or even up to  $\omega$  uniformly. Note that even when  $\text{HOD}^\eta$  exists, there seems little reason to expect that it is necessarily a definable class, even when  $\eta$  is definable or comparatively small, such as  $\eta = \omega$ , although in most of the cases where we know  $\text{HOD}^\eta$  exists it is because we do in fact have a uniform definition. We now generalize the Kunen inconsistency and theorem 11 to the case of embeddings from  $V$  to any  $\text{HOD}^\eta$  or its eventually stationary correct submodels.

**Corollary 16.** *If  $\text{HOD}^\eta$  exists, then there is no nontrivial elementary embedding  $j : V \rightarrow \text{HOD}^\eta$ . More generally, if  $\text{HOD}^\eta$  exists and  $M \subseteq \text{HOD}^\eta$  is eventually stationary correct relative to  $\text{HOD}^\eta$ , then there is*

*no nontrivial elementary embedding  $j : V \rightarrow M$ . Indeed, no such embeddings exist in any set-forcing extension  $V[G]$ .*

*Proof.* For the easy case, if  $j : V \rightarrow \text{HOD}^\eta$ , then since  $\text{HOD}^\eta \subseteq \text{HOD}$ , it follows by corollary 13 that  $V = \text{HOD}$  and hence  $V = \text{HOD}^\eta$ . The existence of such a  $j$  is now ruled out by theorem 10. More generally, suppose that  $j : V \rightarrow M \subseteq \text{HOD}^\eta$  is a nontrivial elementary embedding that is a class in some set-forcing extension  $V[G]$ , and that  $M$  is eventually stationary correct to  $\text{HOD}^\eta$ . We may assume by further collapse forcing if necessary that the forcing giving rise to  $V[G]$  is ordinal definable and almost homogeneous. Since  $V$  is eventually stationary correct to  $V[G]$ , it follows by theorem 14 applied to  $j$  in  $V[G]$  that every element of  $V$  is definable in  $V[G]$  from parameters in  $M$ , which are all in  $\text{HOD}^\eta$ , and so  $V \subseteq \text{HOD}^\eta$ , which implies  $V = \text{HOD}^\eta$ . The existence of  $j$  is now ruled out by theorem 10 applied in  $V[G]$ , since  $M$  is eventually stationary correct in  $\text{HOD}^\eta$ .  $\square$

We should now like to generalize some of the previous results from  $\text{HOD}$  to the case of the *generic*  $\text{HOD}$ , denoted  $\text{gHOD}$ , defined to be the intersection of the  $\text{HOD}$ s of all set-forcing extensions. It suffices to consider only forcing notions of the form  $\text{Coll}(\omega, \theta)$ , as these absorb all other forcing and do so while not enlarging  $\text{HOD}$ , because they are almost-homogeneous and ordinal definable. The  $\text{gHOD}$  is a definable transitive proper class model of ZFC and invariant by set forcing. The generic  $\text{HOD}$  was introduced by Gunter Fuchs [Fuc08, p. 298] and further explored in [FHR], where results show that it is consistent that the  $\text{gHOD}$  is far smaller than  $\text{HOD}$  and also smaller than the mantle, the intersection of all set-forcing ground models of  $V$ . For any class  $A$ , the class  $\text{HOD}(A)$  consists of the sets that are hereditarily definable using ordinal parameters or parameters in  $A$ . The class  $\text{gHOD}(A)$  is the intersection of  $\text{HOD}(A)^{V[G]}$  over all set-forcing extensions  $V[G]$ .

**Corollary 17.** *Suppose that  $j : M \rightarrow N$  is a nontrivial elementary embedding between inner models  $M$  and  $N$  of ZFC and that  $M$  is eventually stationary correct to  $V$ . Then  $M \subseteq \text{gHOD}(N)$  and furthermore  $j \restriction A \in \text{gHOD}(N)$  for any  $A \in M$ .*

*Proof.* If  $j : M \rightarrow N$  is a nontrivial elementary embedding between inner models  $M$  and  $N$  and  $M$  is eventually stationary correct to  $V$ , then it is also eventually stationary correct to  $V[G]$  for any set-forcing extension of  $V$ , and so by theorem 14 applied in  $V[G]$ , it follows that  $M \subseteq \text{HOD}(N)^{V[G]}$ . But  $\text{gHOD}(N)$  in  $V$  is the intersection of all such  $\text{HOD}(N)^{V[G]}$ , so the proof is complete.  $\square$

**Corollary 18.** *If  $j : V \rightarrow M$  and  $M \subseteq \text{gHOD}$ , then  $V = \text{gHOD}$ .*

*Proof.* Since  $\text{gHOD} \subseteq \text{HOD}$  and this remains true in any set-forcing extension, and since  $V$  is eventually stationary correct to every set-forcing extension  $V[G]$ , it follows by theorem 14 that  $V \subseteq \text{HOD}^{V[G]}$  for all such extensions, and this implies  $V \subseteq \text{gHOD}$  and hence  $V = \text{gHOD}$ , as desired.  $\square$

The ideas apply to the iterated  $\text{gHOD}$  construction as well. We use the phrase “ $\text{gHOD}^\eta$  exists” to mean that  $\eta$  is an ordinal and there is a class  $H$ , whose slices  $H^\alpha$  are all models of ZF and constitute a uniform presentation of the  $\text{gHOD}^\alpha$  for  $\alpha \leq \eta$ , having the correct base case  $H^0 = V$ , successor step  $H^{\alpha+1} = \text{gHOD}^{H^\alpha}$  and limit case  $H^\gamma = \bigcap_{\alpha < \gamma} H^\alpha$  (for limit ordinals  $\gamma$ ).

**Corollary 19.** *There is no nontrivial elementary embedding*

$$j : V \rightarrow \text{gHOD}.$$

*If  $\text{gHOD}^\eta$  exists, then there is no nontrivial elementary embedding*

$$j : V \rightarrow \text{gHOD}^\eta.$$

*More generally, if  $\text{gHOD}^\eta$  exists and  $M \subseteq \text{gHOD}^\eta$  is eventually stationary correct relative to  $\text{gHOD}^\eta$ , then there is no nontrivial elementary embedding  $j : V \rightarrow M$ .*

*Proof.* If  $j : V \rightarrow M \subseteq \text{gHOD}^\eta$ , then since  $M \subseteq \text{gHOD}$  it follows by corollary 18 that  $V = \text{gHOD}$  and so  $V = \text{gHOD}^\eta$ . The embedding is now ruled out by theorem 10.  $\square$

We note as before that since AC holds automatically in  $\text{gHOD}$ , we need not assume AC in  $V$  when ruling out a nontrivial elementary embedding  $j : V \rightarrow \text{gHOD}$ , as this assumption follows by elementarity. A similar observation holds for  $j : V \rightarrow \text{gHOD}^{\eta+1}$ .

The results of corollaries 16 and 19 generalize to the case of  $\text{HOD}[A]$  and  $\text{gHOD}[A]$ , where  $A$  is any class, as well to as their iterates and eventually stationary correct submodels. Namely, the class  $\text{HOD}[A]$  is the class of all sets hereditarily definable from ordinal parameters and using  $A$  as a predicate, and  $\langle \text{HOD}[A], \in, A \rangle$  is a model of  $\text{ZFC}(A)$ . Iterating this idea, we define that  $\text{HOD}[A]^\eta$  exists if there is uniform presentation class  $H$  whose slices obey the desired defining properties, so that  $H^0 = V$ ,  $H^{\alpha+1} = \text{HOD}[A]^{H^\alpha}$  and  $H^\gamma = \bigcap_{\alpha < \gamma} H^\alpha$  for limit  $\gamma$ . A similar definition applies to  $\text{gHOD}[A]$  and  $\text{gHOD}[A]^\eta$ .

**Corollary 20.** *If  $A$  is any class and  $\text{HOD}[A]^\eta$  exists, then there is no nontrivial elementary embedding  $j : V \rightarrow \text{HOD}[A]^\eta$ , and indeed, no nontrivial elementary embedding  $j : V \rightarrow M$  for any class  $M \subseteq \text{HOD}[A]^\eta$  that is eventually stationary correct with respect to  $\text{HOD}[A]^\eta$ .*

*Similarly, if  $\text{gHOD}[A]^\eta$  exists, then there is no nontrivial elementary embedding  $j : V \rightarrow M$  for any class  $M \subseteq \text{gHOD}[A]^\eta$  that is eventually stationary correct in  $\text{gHOD}[A]^\eta$ . And no such embeddings can be found in any set-forcing extension  $V[G]$ .*

The proof proceeds simply by carrying the class  $A$  all the way through the earlier proofs. For example, if  $j : V \rightarrow \text{HOD}[A]$ , then it follows by theorem 12 that every element of  $V$  is definable using parameters from  $\text{HOD}[A]$ , and so  $V = \text{HOD}[A]$ , which now violates theorem 10.

The results in this section all rule out nontrivial  $j : V \rightarrow M$  for various definable classes  $M$ , usually with  $M \subseteq \text{HOD}$ . Perhaps it is tempting to hope for a completely general result ruling out  $j : V \rightarrow M$  for any definable class  $M$ , or perhaps for any definable class  $M \subseteq \text{HOD}$ . But alas, such a result is not generally true, assuming the consistency of a measurable cardinal. To see this, observe that in the canonical inner model  $V = L[\mu]$  with one measurable cardinal, there is a unique normal measure  $\mu$  on a unique measurable cardinal. The ultrapower  $j : L[\mu] \rightarrow L[j(\mu)]$  by this normal measure is consequently a parameter-free definable embedding of  $V = L[\mu]$  into a definable class  $L[j(\mu)]$ , and the model satisfies  $V = \text{HOD}$ . So we should not hope to rule out all nontrivial  $j : V \rightarrow M$  into a definable class  $M \subseteq \text{HOD}$ . Nevertheless, in theorem 28 we shall rule out nontrivial embeddings in the converse direction  $j : M \rightarrow V$ , from any definable class  $M$  to  $V$ .

## 5. EMBEDDINGS FROM DEFINABLE CLASS MODELS INTO $V$

In this section, we rule out elementary embeddings in the converse direction, from  $\text{HOD} \rightarrow V$  and from other definable classes into  $V$ , and the arguments will have a very different character than theorem 11. In particular, the argument ruling out  $j : \text{HOD} \rightarrow V$  will not rely on any result in infinite combinatorics, such as the stationary partition theorem. Instead, we shall extend the embedding  $\text{HOD} \rightarrow V$  into an infinite inverse system of embeddings

$$\dots \longrightarrow \text{HOD}^n \longrightarrow \dots \longrightarrow \text{HOD}^2 \longrightarrow \text{HOD} \longrightarrow V,$$

and then analyze the nature of the inverse limit, ultimately relying on the fact that  $\text{HOD}$  has a definable well-order in  $V$ . The overall argument is soft, simply making use of the inverse system, but the details run into a subtle metamathematical issue, which we resolve, concerning the extent to which we may treat the inverse system uniformly in  $n$ , since as we have mentioned the  $\text{HOD}^n$  sequence is not generally uniformly definable.

After proving in theorem 23 that there is no nontrivial elementary embedding from HOD to  $V$ , we mount a generalization of the methods, culminating in theorem 28, asserting that if  $M$  is any definable class, then there is no nontrivial elementary embedding  $j : M \rightarrow V$ . Few of the theorems in this section require the axiom of choice, although in several cases this is simply because AC follows from the other assumptions.

We begin with the observation that there is a very easy proof of the Kunen inconsistency in the case  $V = \text{HOD}$ .

**Theorem 21.** *If  $V = \text{HOD}$ , then there is no nontrivial elementary embedding  $j : V \rightarrow V$ .*

*Proof.* Let  $\kappa$  be the critical point of  $j$ , and define the usual critical sequence  $\kappa_{n+1} = j(\kappa_n)$ , with  $\kappa_0 = \kappa$  and  $\lambda = \sup\langle \kappa_n \mid n < \omega \rangle$ . By applying  $j$  to the critical sequence, it follows easily that  $j(\lambda) = \lambda$ . Since  $V = \text{HOD}$ , there is a HOD-least  $\omega$ -sequence  $s = \langle \alpha_n \mid n < \omega \rangle$  such that  $\lambda = \sup\langle \alpha_n \mid n < \omega \rangle$ . Since  $s$  is definable from  $\lambda$  and  $j(\lambda) = \lambda$ , it must be that  $j(s) = s$ . In particular,  $j(\alpha_n) = \alpha_n$  for every  $n < \omega$ . But there are no fixed points of  $j$  between  $\kappa$  and  $\lambda$ , so  $s$  is not cofinal in  $\lambda$  after all, a contradiction.  $\square$

We suggest that this argument may be the simplest proof that there is no nontrivial  $j : V \rightarrow L$ . Also, we note that one doesn't need the full  $V = \text{HOD}$  in theorem 21, but only a definable well-ordering of  $[\lambda]^\omega$ , and this observation allows us to transfer the argument to the  $I_1(\kappa)$  context.

**Corollary 22.** *Assume only ZF and suppose that  $j : V_{\lambda+1} \rightarrow V_{\lambda+1}$  is a nontrivial elementary embedding. Then there is no well-ordering of  $[\lambda]^\omega$  definable in  $V_{\lambda+1}$ .*

*Proof.* The same argument as in theorem 21 works here, except that we choose  $s$  by using the definable well-ordering of  $[\lambda]^\omega$ .  $\square$

Let us now rule out the possibility of a nontrivial elementary embedding  $j : \text{HOD} \rightarrow V$ . After this, we shall generalize to other inner models.

**Theorem 23.** *There is no nontrivial elementary embedding*

$$j : \text{HOD} \rightarrow V.$$

*Proof.* Assume to the contrary that there is such a nontrivial elementary embedding  $j : \text{HOD} \rightarrow V$ . As before, we needn't assume AC explicitly since our hypothesis implies that AC holds in  $V$ , since it holds in HOD and  $j$  is elementary. We begin by constructing from  $j$  a



uniform presentation of the classes  $\text{HOD}^n$  for  $n < \omega$ . In order to do so, we would like to iterate  $j$ , and we remark that for this kind of embedding, where the domain is a proper subclass of the codomain, one does not iterate the embedding in the usual forward direction, since it can happen for a set  $x$  that  $j(x)$  is no longer in the domain of  $j$ , leaving  $j^2(x)$  undefined. Rather, one should build an inverse system, iterating the embedding in the reverse direction, with the domains becoming successively smaller. In order to do this, define that  $x$  is in the domain of  $j^n$  if it is possible to apply  $j$  successively  $n$  times to  $x$ ; that is, if there is a sequence  $\langle x_0, \dots, x_n \rangle$  such that  $x_0 = x$  and  $x_{n+1} = j(x_n)$ . It follows that  $x \in \text{dom}(j^{n+1}) \iff j(x) \in \text{dom}(j^n)$ , where we consider  $j^0$  as the universal identity function. Let  $H$  be the class  $\{ (x, n) \mid x \in \text{dom}(j^n) \}$ . In short, we define a class  $H$  by specifying its slices as  $H^n = \text{dom}(j^n)$ . We will show that  $H$  is a uniform presentation of the  $\text{HOD}^n = H^n$ . Clearly,  $H^0 = V$  and  $H^1 = \text{HOD}$ , and we are on our way; it remains to prove  $H^{n+1} = \text{HOD}^{H^n}$ , which we now do by induction. This argument makes subtle but critical use of the fact, due to lemmas 2 and 3, that the statements of the following claim are each expressible by single NGBC assertions in the class parameter  $j$ , with natural number parameter  $n$ .<sup>4</sup>

**Claim 23.1.** *For all  $n \in \omega$ ,*

- (1)  $H^{n+1} = \text{HOD}^{H^n}$ .
- (2)  $H^n$  and  $H^{n+1}$  are transitive, proper class models of ZFC.
- (3)  $j \upharpoonright H^{n+1} : H^{n+1} \rightarrow H^n$  is elementary.

*Proof.* Note that we need statement 2 to hold even in order for statement 3 to be first-order expressible, since Gaifman's lemma 2 as we have stated it applies only to transitive models of ZF.<sup>5</sup> The case  $n = 0$

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<sup>4</sup>The subtle point is that one cannot generally prove a *scheme* of statements  $\varphi_i(n)$  by induction on  $n$ , since if the scheme is not expressible, we are not able in NGBC to form the set of  $n$  where it fails and thus may be unable to find the least  $n$  where it fails.

<sup>5</sup>For example, recent work of Gitman, Hamkins and Johnstone [GHJ] shows that Gaifman's theorem can fail for transitive proper class models of the version of ZFC without Powerset using Replacement without Collection, although it does apply to  $\text{ZFC}^-$  models when this theory includes Collection.

is immediate. Assume inductively that the claim holds for  $n$ , and consider  $n + 1$ . We start with statement 1. For all  $x \in H^{n+1}$ , we have

$$\begin{aligned}
 x &\in \text{HOD}^{H^{n+1}} \\
 &\iff H^{n+1} \models x \in \text{HOD} \\
 &\iff H^n \models j(x) \in \text{HOD} && \text{by inductive hypothesis 3} \\
 &\iff j(x) \in H^{n+1} && \text{by inductive hypothesis 1} \\
 &\iff x \in H^{n+2} && \text{by definition of } H.
 \end{aligned}$$

So statement 1 is proven for  $n + 1$ .

Next, we prove statement 2. By the inductive hypothesis,  $H^{n+1}$  is a ZFC model and therefore satisfies the sentence stating that  $\text{HOD}$  is a transitive, proper class ZFC model. Since we just proved that  $H^{n+2} = \text{HOD}^{H^{n+1}}$ , it follows that  $H^{n+2}$  is a transitive, proper class ZFC model.

Finally, we prove statement 3. First, note by the definition of  $H$  that  $j$  maps  $H^{n+2}$  into  $H^{n+1}$ . Since  $H^{n+2}$  is the  $\text{HOD}$  of  $H^{n+1}$  and  $H^{n+1}$  is the  $\text{HOD}$  of  $H^n$ , and we know by the inductive hypothesis that  $j \upharpoonright H^{n+1} : H^{n+1} \rightarrow H^n$  is elementary, it follows that  $j \upharpoonright H^{n+2} : H^{n+2} \rightarrow H^{n+1}$  is elementary, since this is the restriction of an elementary embedding to the definable transitive class  $\text{HOD}$  of the domain and its corresponding codomain. So statement 3 holds and the proof of claim 23.1 is complete.  $\square$

We may now freely refer to  $\text{HOD}^n$  uniformly in  $n$ . The claim shows moreover that we have an entire inverse system of embeddings

$$\dots \longrightarrow \text{HOD}^n \longrightarrow \dots \longrightarrow \text{HOD}^2 \longrightarrow \text{HOD} \longrightarrow V,$$

where the embedding at each step is the appropriate restriction of  $j$ . By composing,  $j^n : \text{HOD}^n \rightarrow V$  is elementary.

Let  $<^{n+1}$  be the canonical well-ordering of  $\text{HOD}^{n+1}$  definable in  $\text{HOD}^n$ . Let  $<^0 = j(<^1)$ , in the sense of applying  $j$  to a class, meaning  $<^0 = \bigcup_{\alpha \in \text{ORD}} j(<^1 \cap V_\alpha)$ . Note that our uniform presentation of the  $\text{HOD}^n$ 's allows us to define the  $<^{n+1}$  uniformly in  $n$ . Since  $j$  is elementary from each  $\text{HOD}^n$  to  $\text{HOD}^{n-1}$ , it follows that  $j(<^{n+1}) = <^n$  for all  $n < \omega$ , again in the sense of applying  $j$  to a class.

The key concept of this proof is the definition that a sequence  $\vec{x} = \langle x_n \mid n < \omega \rangle$  is *j-coherent*, if  $j(x_{n+1}) = x_n$  for all  $n < \omega$ . Such sequences arise naturally in the inverse limit of the system of embeddings above. We shall now derive a contradiction by showing first, that every *j-coherent* sequence is constant, and second, that there is a *j-coherent* sequence that is not constant.

We show first that every  $j$ -coherent sequence is constant, having the form  $\langle x, x, x, \dots \rangle$  for a fixed point  $x = j(x)$ . To see this, suppose that  $\vec{x} = \langle x_n \mid n < \omega \rangle$  is a nonconstant  $j$ -coherent sequence, where  $x_0$  has minimal possible  $\in$ -rank, which we denote  $\text{rank}(x_0)$ . Since  $j(\text{rank}(x_{n+1})) = \text{rank}(x_n)$  by elementarity, it follows that  $\text{rank}(x_{n+1}) \leq \text{rank}(x_n)$ . If  $\text{rank}(x_1) < \text{rank}(x_0)$ , then since the sequence is  $j$ -coherent, we may apply the inverse of  $j^n$  and see that  $\text{rank}(x_{n+1}) < \text{rank}(x_n)$ , thereby producing an infinite descending sequence of ordinals, which is impossible. So it must be that all the  $x_n$  have the same rank. It is similarly easy to see that  $x_{n+1} \neq x_n$ , since otherwise the entire sequence would be constant, contrary to our assumption. Let  $a_n$  be the  $<^n$ -least element of the symmetric difference  $x_{n+1} \Delta x_n$ , which makes sense because these two sets are distinct elements of  $\text{HOD}^n$ . By the  $j$ -coherence of the well-orderings  $<^n$  and of  $\vec{x}$ , it follows that  $j(a_{n+1}) = a_n$ , and so  $\vec{a} = \langle a_n \mid n < \omega \rangle$  is  $j$ -coherent. Note that  $a_1 \neq a_0$ , since if  $a_0 \in x_1 \setminus x_0$  then  $a_1 \in x_2 \setminus x_1$  by elementarity, and otherwise  $a_0 \in x_0 \setminus x_1$ , leading to  $a_1 \in x_1 \setminus x_2$ , which makes  $a_0 = a_1$  impossible in either case. Finally, since  $a_0 \in x_0 \Delta x_1$ , the rank of  $a_0$  is smaller than the rank of  $x_0$  (which is equal to the rank of  $x_1$ ). This contradicts our assumption that  $\vec{x}$  was a minimal counterexample, and so we have established that all  $j$ -coherent sequences are trivial.

We now obtain a contradiction by producing a nontrivial  $j$ -coherent sequence. By theorem 21, we know that  $\text{HOD} \subsetneq V$ , and consequently  $\text{HOD}^{n+1} \subsetneq \text{HOD}^n$  by elementarity. Let  $s_n$  be the  $<^n$ -least element of  $\text{HOD}^n \setminus \text{HOD}^{n+1}$ . It follows by the  $j$ -coherence of the relations  $<^n$  that  $j(s_{n+1}) = s_n$ , and so this sequence is  $j$ -coherent. Since  $s_0 \in V \setminus \text{HOD}$  and  $s_1 \in \text{HOD} \setminus \text{HOD}^2$ , it follows that  $s_0 \neq s_1$ , and so this sequence is not constant, a contradiction.  $\square$

We shall now generalize theorem 23 to other definable classes by first establishing the following fundamental fact. A class  $A$  is *b-definable* in a transitive inner model  $M$  if there is a formula  $\varphi$  such that  $A = \{x \in M \mid M \models \varphi(x, b)\}$ . If  $N$  is another model containing  $b$ , we may relativize the definition to obtain  $A^N = \{x \in N \mid N \models \varphi(x, b)\}$ . The transitive closure of a class is the smallest transitive class containing all elements of that class.

**Theorem 24.** *Do not assume AC. Suppose that  $j : M \rightarrow N$  is an elementary embedding of inner models  $M \subseteq N$  of ZF, where  $M$  is  $b$ -definable in  $N$  with a parameter  $b$  fixed by  $j$ . Suppose that  $A$  is a  $b$ -definable class (or set) in  $N$  and that the transitive closure of  $A$  has a  $b$ -definable well ordering in  $N$ ; or equivalently,  $A$  is a  $b$ -definable class in  $N$ , and  $A \subseteq \text{HOD}[b]^N$ . Then  $A^M = A^N$ .*

*Proof.* First, we note that given  $A$  is  $b$ -definable in  $N$ , the inclusion  $A \subseteq \text{HOD}[b]^N$  is equivalent to the existence of a  $b$ -definable well-ordering of  $\text{tcl}(A)$  in  $N$ . In one direction, if  $A \subseteq \text{HOD}[b]^N$ , then the canonical  $\text{HOD}[b]^N$  ordering orders  $\text{tcl}(A)$ . Conversely, if  $\text{tcl}(A)$  has a  $b$ -definable well-ordering, then since  $A$  is  $b$ -definable, it follows for each ordinal  $\alpha$  that the  $\alpha$ -th element of  $A$  under this well-ordering is definable from  $b$  and  $\alpha$  in  $N$ , and so  $A \subseteq \text{HOD}[b]^N$ .

The theorem is formalized as an NGB theorem scheme, asserting of each pair of formulas  $\psi$  and  $\varphi$  that if they define  $M$  and  $A$  respectively in  $N$  in the way described, then the conclusion holds. To begin the proof, suppose that  $j : M \rightarrow N$  is as described in the hypothesis of the theorem. Let  $M^0 = N$  and  $M^n = \text{dom}(j^n)$ , as defined in the proof of theorem 23, so that we have a uniform presentation of these classes. In this notation, the original embedding is  $j : M^1 \rightarrow M^0$ . Note that because  $b = j(b)$ , it follows that  $b \in M^n$  for every  $n$ .

**Claim 24.1.** *For all  $n \in \omega$ ,*

- (1)  $M^{n+1}$  is the  $M$  of  $M^n$ , using the same definition of  $M$  as in  $N$ , with the same parameter  $b$ .
- (2)  $M^n$  and  $M^{n+1}$  are transitive, proper class models of ZF.
- (3)  $j \upharpoonright M^{n+1} : M^{n+1} \rightarrow M^n$  is elementary.

The proof is by induction on  $n$ , and follows the proof of claim 23.1 by substituting  $M$  in place of  $\text{HOD}$  and using the fact that  $j(b) = b$ . Thus, the definable classes  $M^n$  essentially form a  $j$ -coherent sequence, and we omit the details. The claim leads to the inverse system

$$\dots \longrightarrow M^n \longrightarrow \dots \longrightarrow M^2 \longrightarrow M^1 \longrightarrow M^0,$$

where the embedding at each step is the appropriate restriction of  $j$ , and the final step is the full original embedding  $j : M \rightarrow N$ .

Let us denote by  $<^0$  the hypothesized  $b$ -definable well-order of the transitive closure  $\text{tcl}(A)$  in  $N$ . For each natural number  $n$ , let  $<^n$  be the corresponding well-ordering of  $\text{tcl}(A^{M^n})$  defined in  $M^n$  using the same formula and parameter  $b$ . The fact that this definition is indeed a well-order of  $\text{tcl}(A^{M^n})$  in  $M^n$  follows by the elementarity of  $j$  and the fact that  $j(b) = b$ . Note that the  $<^n$  can be uniformly presented with respect to  $n$ , using our uniform presentation of the classes  $M^n$ . Furthermore, since the  $<^n$  are all defined in  $M^n$  by the same formula and the parameter is fixed by  $j$ , it follows by elementarity that  $j(<^{n+1}) = <^n$ , in the sense of applying  $j$  to a class.

Using the  $j$ -coherent concept of theorem 23, we show that every  $j$ -coherent sequence  $\vec{x} = \langle x_n \mid n \in \omega \rangle$  with  $x_0 \in \text{tcl}(A^{M^0})$  is constant. If not, then let  $\vec{x}$  be a nonconstant  $j$ -coherent sequence with  $x_0 \in$

$\text{tcl}(A^{M^0})$ , such that  $x_0$  has minimal rank among all such sequences. By elementarity, it follows that  $x_n \in \text{tcl}(A^{M^n})$  for every  $n < \omega$ . Since  $\vec{x}$  is not constant, we must have  $x_0 \neq x_1$ . Suppose, for a first case, that  $x_0 \setminus x_1$  is nonempty. It follows by elementarity that  $x_n \setminus x_{n+1}$  is nonempty for all  $n$ , and we may let  $a_n$  be the  $<^n$ -least element of  $x_n \setminus x_{n+1}$ . Note that  $a_n \in \text{tcl}(A^{M^n})$ . By the  $j$ -coherence of  $<^n$  and  $\vec{x}$ , it follows that  $j(a_{n+1}) = a_n$ , and so  $\vec{a} = \langle a_n \mid n < \omega \rangle$  is  $j$ -coherent. Note that  $\vec{a}$  is not constant, since  $a_0 \in x_0 \setminus x_1$  and  $a_1 \in x_1 \setminus x_2$ , and  $a_0 \in x_0$  implies that  $a_0$  is in  $\text{tcl}(A^{M^0})$  with strictly lower rank than  $x_0$ . The existence of  $\vec{a}$  therefore contradicts our minimality assumption on  $\vec{x}$  and  $x_0$ . For the remaining case,  $x_1 \setminus x_0$  is nonempty. By elementarity,  $x_{n+1} \setminus x_n$  is nonempty for every  $n$ , and we may let  $a_{n+1}$  be the  $<^{n+1}$ -least element of  $x_{n+1} \setminus x_n$  and also define  $a_0 = j(a_1)$ . By the uniformity of the definition, it again follows that  $j(a_{n+1}) = a_n$ , and so  $\vec{a} = \langle a_n \mid n < \omega \rangle$  is  $j$ -coherent. It is not constant since  $a_1 \in x_1 \setminus x_0$ , whilst  $a_2 \in x_2 \setminus x_1$ , and from  $a_1 \in x_1$  we deduce that  $a_0 \in x_0 \in \text{tcl}(A^{M^0})$  and so  $a_0$  is in  $\text{tcl}(A^{M^0})$  with strictly lower rank than  $x_0$ , again contradicting our minimality assumption.

Finally, under the assumption that  $A^M \neq A^N$ , we construct a non-constant  $j$ -coherent sequence,  $\vec{s} = \langle s_n \mid n < \omega \rangle$ , with  $s_0 \in \text{tcl}(A^{M^0})$ , thereby contradicting the fact just established that all such sequences are constant. If it happens that  $A^{M^0} \setminus A^{M^1}$  is nonempty, then let  $s_0$  be the  $<^0$ -least element of this difference class. Since this is definable in  $M^0$  from parameter  $b = j(b)$ , it extends naturally to a  $j$ -coherent sequence  $\vec{s} = \langle s_n \mid n < \omega \rangle$ , where  $s_n$  is the  $<^n$ -least element of  $A^{M^n} \setminus A^{M^{n+1}}$ . This sequence is not constant because  $s_0 \in A^{M^0} \setminus A^{M^1}$  but  $s_1 \in A^{M^1} \setminus A^{M^2}$ , and it has  $s_0 \in A^{M^0}$ , as desired. In the remaining case,  $A^{M^1} \setminus A^{M^0}$  is nonempty. Let  $s_{n+1}$  be the  $<^{n+1}$ -least element of  $A^{M^{n+1}} \setminus A^{M^n}$ , and let  $s_0 = j(s_1)$ . By the uniformity of these definitions, it follows that  $j(s_{n+1}) = s_n$ , and so  $\vec{s} = \langle s_n \mid n < \omega \rangle$  is  $j$ -coherent. But it is not constant, because  $s_1 \in A^{M^1} \setminus A^{M^0}$  whilst  $s_2 \in A^{M^2} \setminus A^{M^1}$ , and since  $s_1 \in A^{M^1}$ , we have  $s_0 \in A^{M_0} \subseteq \text{tcl}(A^{M^0})$ . So once again, we have contradicted the fact established in the previous paragraph, and the proof is complete.  $\square$

**Corollary 25.** *If  $j : M \rightarrow N$  is an elementary embedding of inner models  $M \subseteq N$  of ZF, where  $M$  is  $b$ -definable in  $N$  using a parameter  $b$  fixed by  $j$ , then  $M$  and  $N$  have*

- (1) *the same cardinals and the same cofinality function,*
- (2) *the same continuum function,*
- (3) *the same  $\diamond_\kappa^*$  pattern and*

- (4) *the same large cardinals of any particular kind.*
- (5) *Furthermore,  $\text{HOD}^M = \text{HOD}^N$  and  $\text{gHOD}^M = \text{gHOD}^N$  and more.*

*Proof.* As in theorem 24, we do not need to assume AC. The corollary follows immediately from the theorem, because in each case we have a definable class having a definable well-order on its transitive closure. For example, if  $A$  is the class of cardinals in  $N$ , then this is definable in  $N$  and the transitive closure is the class  $\text{ORD}^N$ , which certainly has a definable well-order in  $N$ . So by the theorem,  $A^M = A^N$ , and so  $M$  and  $N$  have the same cardinals. Similarly, we can let  $A$  be the graph of the cofinality function, or the graph of the continuum function  $\gamma \mapsto 2^\gamma$ , or the class of cardinals  $\kappa$  for which  $\diamond_\kappa^*$ , or the class of measurable cardinals, or the class of supercompact cardinals or what have you. In each case, the class is definable and has a definable well-order on the transitive closure, and so the theorem implies that the class has the same extension in  $M$  as in  $N$ . Similarly, the case of  $\text{HOD}$  and  $\text{gHOD}$  are definable transitive classes with a definable well-order, and so by the theorem have the same extension in  $M$  and  $N$ . The proof method clearly applies to many other definable classes.  $\square$

**Corollary 26.** *Suppose that  $M \subsetneq N$  are inner models of ZF, and that  $M$  is definable in  $N$  and  $M \subseteq \text{HOD}^N$ . Then there is no nontrivial elementary embedding  $j : M \rightarrow N$ .*

*Proof.* If there were such a  $j : M \rightarrow N$ , then by corollary 25, it follows that  $\text{HOD}^M = \text{HOD}^N$ . In this case, it follows that  $M \subseteq \text{HOD}^N = \text{HOD}^M$  and so  $M = \text{HOD}^M$ , and consequently  $N = \text{HOD}^N$  and so  $M = N$ , contrary to assumption.  $\square$

It follows immediately that there is no nontrivial elementary embedding  $j : \text{HOD}^2 \rightarrow \text{HOD}$ , if these models are different, and indeed, there is no  $j : \text{HOD}^n \rightarrow \text{HOD}^m$  for any  $m < n < \omega$ , if the models are different. More generally, if  $\text{HOD}^\eta$  exists and is definable in  $\text{HOD}^\xi$  for some  $\xi < \eta$ , and  $\text{HOD}^\eta \neq \text{HOD}^\xi$ , then there is no nontrivial  $j : \text{HOD}^\eta \rightarrow \text{HOD}^\xi$ . Similarly, it follows immediately from corollary 26 that there is no nontrivial  $j : \text{gHOD}^2 \rightarrow \text{gHOD}$  and no nontrivial  $j : \text{gHOD}^n \rightarrow \text{gHOD}^m$  for any  $m < n < \omega$ , provided these models differ, and indeed, if  $\text{gHOD}^\eta$  exists and is definable in  $\text{gHOD}^\xi$  for some  $\xi < \eta$ , then there is no nontrivial  $j : \text{gHOD}^\eta \rightarrow \text{gHOD}^\xi$ , provided the models differ. In each case, we would have definable  $M \subseteq \text{HOD}^N$ , and so corollary 26 rules out such  $j$ .

Theorem 24 can be applied to transitive *set* models  $M$  and  $N$ , where  $M \subseteq N$  is definable, they have the same ordinals and there is a cofinal

elementary embedding  $j : M \rightarrow N$ , and this situation allows for several simplifications. In this case, one can dispense with many of the metamathematical concerns about uniform presentations, since one can perform the induction in the ambient set theoretic background, where  $j$  would now be a set. One still seems to need that  $M$  and  $N$  are well-founded in order to pick  $x_0$  of minimal rank, although this could also be possible even when the models are ill-founded, as long as the  $M^n$  are uniformly presented in  $N$  and  $j$  is amenable to  $N$ , although such a situation would be similar simply to applying the current theorem inside  $N$ .

We also briefly note that the conclusion of theorem 24 applies even in the case that the class is defined using an ordinal parameter that is not necessarily fixed by  $j$ .

**Corollary 27.** *Suppose that  $j : M \rightarrow N$  is an elementary embedding of inner models  $M \subseteq N$  of ZF, where  $M$  is  $b$ -definable in  $N$  with a parameter  $b$  fixed by  $j$ . Suppose that  $B \subseteq \text{HOD}[b]^N$  is  $(b, \beta)$ -definable in  $N$  for some ordinal  $\beta$ . Then  $B^M = B^N$ .*

*Proof.* Again, we do not need AC here. Since  $b$  and  $\beta$  are in both  $M$  and  $N$ , it makes sense to consider  $B^M$ , defined using the same definition as in  $N$  and the same parameters, so that  $x \in B \iff \varphi(x, b, \beta)$  in either model. In  $N$ , define the class  $A = \{ \langle x, \alpha \rangle \in \text{HOD}[b] \mid \varphi(x, b, \alpha) \}$ . This class is  $b$ -definable in  $N$  and contained in  $\text{HOD}[b]^N$ . Thus, by theorem 24, it follows that  $A^M = A^N$ . But the class  $B$  is simply the  $\beta^{\text{th}}$  slice of  $A$ , and so it follows that  $B^M = B^N$ .  $\square$

The previous corollaries will now allow us to apply the stationary partition argument in a range of additional situations. For example, we find the following consequence striking, and it implies many of the other results we have discussed.

**Theorem 28.** *If  $M$  is a definable transitive class in  $V$ , then there is no nontrivial elementary embedding  $j : M \rightarrow V$ .*

This theorem should be understood as an NGBC theorem scheme, asserting of each possible parameter-free definition of such an  $M$ , that no NGBC class is such an embedding  $j$ . The theorem has theorem 23 as a special case, asserting that there is no  $j : \text{HOD} \rightarrow V$ , simply because  $\text{HOD}$  is definable in  $V$ . But it generalizes to show that there is no  $j : \text{HOD}^n \rightarrow V$  for any natural number  $n$ , no  $j : \text{gHOD} \rightarrow V$  and no  $j : \text{gHOD}^n \rightarrow V$ , as all these classes are definable. Theorem 28 is itself an immediate consequence of the following more general result, simply by taking the case  $N = V$ .

**Theorem 29.** *Without assuming AC in  $V$ , if  $j : M \rightarrow N$  is a nontrivial elementary embedding of inner models  $M \subseteq N$  of ZFC and  $N$  is eventually stationary correct to  $V$ , then  $M$  is not definable in  $N$  from parameters fixed by  $j$ .*

*Proof.* This theorem can be formalized as an NGB scheme, asserting of every formula  $\varphi$  that it is not a definition of  $M$  in  $N$  by parameters fixed by  $j$ , supposing that  $j$ ,  $M$  and  $N$  are as described. Suppose that  $j : M \rightarrow N$  is a nontrivial elementary embedding of inner models  $M \subseteq N$  of ZFC, where  $N$  is stationary correct to  $V$  above  $\theta$ , and suppose towards contradiction that  $M$  is  $b$ -definable in  $N$  for some parameter  $b = j(b)$ . Let  $\kappa$  be the critical point of  $j$ , which exists by lemma 4, and let  $C = \{\beta \mid j \restriction \beta \subseteq \beta\}$  be the class of ordinals closed under  $j$ , which is a closed unbounded class of ordinals. Fix any  $\eta \in C$  of cofinality greater than  $\kappa$  and  $\theta$ , and observe that  $C \cap \eta$  is club in  $\eta$ . Let  $S = (\text{Cof}_\omega \eta)^M$ , which by corollary 25 is the same as  $(\text{Cof}_\omega \eta)^N$ , which is stationary in  $N$  and hence also in  $V$ . Thus, we may find  $\lambda \in S \cap C$  above  $\kappa$  and  $\theta$ . Since  $\text{cof}(\lambda) = \omega$  in  $M$  and  $j \restriction \lambda \subseteq \lambda$ , it follows that  $j(\lambda) = \lambda$ . Since  $M$  and  $N$  have the same cardinals by corollary 25, and  $N$  and  $V$  have the same cardinals above  $\theta$  by the discussion about stationary correctness in section 3, it follows that  $(\lambda^+)^M = (\lambda^+)^N = (\lambda^+)^V$ , a cardinal we unambiguously denote  $\lambda^+$ , and so we conclude  $j(\lambda^+) = \lambda^+$ .

Applying the Ulam-Solovay partition theorem in  $M$ , there is a partition  $\vec{S} = \langle S_\alpha \mid \alpha < \lambda^+ \rangle \in M$  of  $(\text{Cof}_\omega \lambda^+)^M = (\text{Cof}_\omega \lambda^+)^N$  into disjoint sets stationary in  $M$ . Let  $\vec{T} = j(\vec{S})$ . By elementarity and stationary correctness,  $T_\kappa$  is stationary in  $V$ . Since  $C \cap \lambda^+$  is club, there exists  $\beta \in T_\kappa \cap C$ , and  $\beta$  has cofinality  $\omega$  in  $N$  and hence in  $M$ . It follows that  $\beta \in S_\alpha$  for some  $\alpha < \lambda^+$ , and that  $j(\beta) = \beta$ . Therefore,  $\beta \in T_{j(\alpha)} \neq T_\kappa$ , contradicting the disjointness of  $\vec{T}$ .  $\square$

As a quick example, we may immediately deduce the following corollary, an extension of theorem 28 to the case of *generic* embeddings, those that exist as classes in a forcing extension. The corollary follows from theorem 29 with the observation that  $V$  is eventually stationary correct in all of its set-forcing extensions.

**Corollary 30.** *If  $M$  is a definable class in  $V$ , then in no set-forcing extension  $V[G]$  is there a nontrivial elementary embedding  $j : M \rightarrow V$ .*

For example, in any set forcing extension  $V[G]$ , there is no nontrivial generic elementary embedding  $j : \text{HOD} \rightarrow V$ , no nontrivial generic embedding  $j : \text{gHOD} \rightarrow V$  and no nontrivial generic embedding from the Mantle or from the generic Mantle to  $V$ .



In connection with theorem 28, let us mention the very interesting work of Vickers and Welch [VW01, p. 1100], who proved that if ORD is Ramsey, then there is a nontrivial elementary embedding  $j : M \rightarrow V$ , where  $M$  is a transitive inner model of ZFC, and  $j$  is definable from a proper class that exists as the result of the large cardinal assumption. Vickers and Welch note (p. 1090) that  $j$  cannot be definable in the usual sense, and our theorem 28 shows moreover that even the class  $M$  is not definable. On page 1101 of the same paper, they reproduce a proof due to Foreman showing that if  $M$  is any inner model closed under  $\omega$  sequences of ordinals, then (assuming AC) there is no  $j : M \rightarrow V$ . One very interesting part of this proof is the use of an ultrapower construction to obtain a regular fixed point above the critical point.

Let us deduce one final corollary of theorem 24, concerning the situation when AC fails. As we shall mention later in questions 38 and 39, it is not known whether one can prove the Kunen inconsistency without the axiom of choice, that is, whether there can be a nontrivial elementary embedding  $j : V \rightarrow V$  in the  $\neg$ AC context, nor whether there can be a nontrivial elementary embedding  $j : \text{HOD} \rightarrow \text{HOD}$ . Of course, the two questions are related, because if there is  $j : V \rightarrow V$  in a  $\neg$ AC context, then the restriction of this embedding produces a nontrivial elementary embedding  $\text{HOD} \rightarrow \text{HOD}$ . The next corollary improves on the observation by showing that in order to produce a nontrivial  $j : \text{HOD} \rightarrow \text{HOD}$ , it suffices to have a nontrivial elementary embedding  $j : M \rightarrow V$  in the  $\neg$ AC context from a definable  $M$  to  $V$ .

**Corollary 31.** *Do not assume AC. If  $j : M \rightarrow V$  is a nontrivial elementary embedding from a transitive proper class  $M$  that is definable in  $V$  from parameters fixed by  $j$ , then there is a nontrivial elementary embedding from  $\text{HOD}$  to  $\text{HOD}$ .*

*Proof.* By theorem 24, it follows from the assumption that  $\text{HOD}^M = \text{HOD}$ , and so  $j \upharpoonright \text{HOD} : \text{HOD} \rightarrow \text{HOD}$  is the desired embedding. Note that lemma 4 shows that  $j$  must have a critical point, and so this restriction is nontrivial.  $\square$

Although we do not know whether or not there can be nontrivial  $j : \text{HOD} \rightarrow \text{HOD}$ , we merely note that corollary 31 may be a way to produce them.

## 6. THE CASE WHERE $j$ IS DEFINABLE

In this section, we consider the Kunen inconsistency and its generalizations in the special case where the embedding  $j$  is not merely a

class in NGBC set theory, but a first-order definable class. In this special case, many of the results admit of particularly soft proofs, which we shall presently describe, relying neither on any deep combinatorial facts nor on the axiom of choice. These results can be formalized as ZF theorem schemes. Since these soft proofs make essential use of the definability of  $j$ , however, they do not seem to generalize to the corresponding full results concerning embeddings that are NGBC classes and not necessarily definable from parameters. In part for this reason, as we mentioned in section 1, we find that the NGBC interpretation of the Kunen inconsistency seems to provide it a more robust content than the ZFC scheme interpretation concerning only definable embeddings. Nevertheless, because the soft proofs here do not use AC, we seem unable to deduce them directly from the prior results, which do use AC.

When we say that an embedding  $j : M \rightarrow N$  on transitive proper class models  $M$  and  $N$  of ZF is definable in  $V$  using parameter  $p$ , what we mean is that there has been already provided a particular first-order formula  $\varphi(x, y, z)$ , such that  $j(x) = y$  if and only if  $\varphi(x, y, p)$  holds in  $V$ . That is, the relation  $\varphi(\cdot, \cdot, p)$  defines the graph of  $j$ . In particular, the domain  $M = \{x \mid \exists y \varphi(x, y, p)\}$  and the codomain  $N = \bigcup \{y \mid \exists x \varphi(x, y, p)\}$  are also definable classes. (We know that  $N = \bigcup \{y \mid \exists x \varphi(x, y, p)\}$  because  $j$  is cofinal, and each element of  $N$  is an element of some  $V_\alpha^N$ .)

Conversely, we point out now that for a fixed first-order formula  $\varphi$ , the question of whether a parameter  $p$  succeeds in defining such an elementary embedding  $j : M \rightarrow N$  via  $\varphi(\cdot, \cdot, p)$  is expressible as a first-order property of  $p$ . To begin, it is easy to express whether  $\varphi(\cdot, \cdot, p)$  defines a functional relation of its first two arguments. The question of whether the domain  $M$  and codomain  $N$  of the function are transitive proper class models of ZF is expressible by the method of lemma 3, and the question of whether the function is elementary is expressible by the method of lemma 2. Similarly, when  $\varphi(\cdot, \cdot, p)$  does define an elementary embedding, the question of whether this embedding is nontrivial is easily expressible, as is the question of whether the embedding has critical point  $\kappa$ . Putting all this together, for a given formula  $\varphi$  the question whether a parameter  $p$  succeeds in defining via  $\varphi(\cdot, \cdot, p)$  a nontrivial elementary embedding  $j : V \rightarrow V$  is a first-order expressible property of  $p$ . Similarly, for a given formula  $\varphi$ , the collection of ordinals  $\kappa$  which arise as the critical point of a nontrivial elementary embedding  $j : V \rightarrow V$  defined by  $\varphi(\cdot, \cdot, p)$  for some parameter  $p$  is a definable class of ordinals.

These observations are all that are required now to prove the Kunen inconsistency for embeddings that are definable from parameters. This result, which had been part of the folklore in some pockets of the set-theoretic community, was published by Suzuki [Suz99]. The essence of the proof is the classical observation that the concept of being a Reinhardt cardinal, if consistent, cannot be first order expressible, since if  $\kappa$  is the least Reinhardt cardinal, witnessed by  $j : V \rightarrow V$ , then by elementarity  $j(\kappa)$  would also be the least Reinhardt cardinal, contrary to  $\kappa < j(\kappa)$ . Indeed, for the same reason, there can be no consistent first-order property  $\varphi(\kappa)$  implying that  $\kappa$  is Reinhardt.

**Theorem 32** ([Suz99]). *Assume only ZF. There is no nontrivial elementary embedding  $j : V \rightarrow V$  that is definable from parameters.*

*Proof.* This is a theorem scheme, asserting of each formula  $\varphi$  that there is no parameter  $p$  for which  $\varphi(\cdot, \cdot, p)$  defines an elementary embedding  $j : V \rightarrow V$ . Suppose that for some parameter  $p$ , the relation  $\varphi(\cdot, \cdot, p)$  defines a nontrivial elementary embedding  $j : V \rightarrow V$ . We may choose  $p$  so that the critical point  $\kappa$  of this embedding is as small as possible, among all parameters  $w$  for which  $\varphi(\cdot, \cdot, w)$  defines a nontrivial elementary embedding, since as we explained before the theorem, these notions are first order expressible. In particular,  $\kappa$  is definable in  $V$ . Since  $j : V \rightarrow V$  is elementary,  $j(\kappa)$  satisfies the same definition, contradicting the fact that  $\kappa < j(\kappa)$ .  $\square$

The proof of theorem 32 worked by observing that if  $j : V \rightarrow V$  is definable in  $V$ , even with parameters, then the concept of being Reinhardt with respect to that definition for some parameter is first order expressible. If it were consistent, then the least such cardinal  $\kappa$  would be definable, contrary to  $\kappa < j(\kappa)$ . The argument therefore follows exactly the pattern mentioned just before theorem 32 of ruling out any consistent first-order property implying the Reinhardt property.

This method, of obtaining the smallest possible critical point by quantifying over all possible choices of parameter, thereby defining the cardinal without any parameters, applies in many other contexts. We shall now generalize the theorem by considering the possibility of definable embeddings added by forcing.

**Theorem 33.** *Do not assume AC. If  $M \subseteq V[G]$  is a transitive inner model of a set-forcing extension  $V[G]$ , then there is no nontrivial elementary embedding  $j : M \rightarrow V$ , with a critical point, that is definable in  $V[G]$  from parameters.*

*Proof.* Let  $\mathbb{Q}$  be a forcing notion,  $G \subseteq \mathbb{Q}$  generic. Suppose there is such a  $j : M \rightarrow V$  defined in  $V[G]$  by the formula  $\varphi(\cdot, \cdot, b)$ , using parameter

$b \in V[G]$ , having critical point  $\kappa$ . Thus, there is some  $\mathbb{Q}$ -name  $\dot{b}$  and some condition  $q \in \mathbb{Q}$  which forces that  $\varphi(\cdot, \cdot, \dot{b})$  defines a nontrivial elementary embedding from a transitive inner model to  $\check{V}$  with critical point  $\check{\kappa}$ . Since this property is first-order expressible, we may assume without loss of generality that  $\kappa$  is the smallest ordinal for which there is a forcing notion  $\mathbb{Q}$  with a condition  $q \in \mathbb{Q}$  and  $\mathbb{Q}$ -name  $\dot{b}$  for which  $q$  forces this fact about  $\check{\kappa}$ . Thus,  $\kappa$  is definable in  $V$  without parameters. Since  $j : M \rightarrow V$  is elementary, this implies that  $\kappa$  must be in the range of  $j$ , contrary to  $\kappa$  being the critical point.  $\square$

The definable-embedding analogues of theorems 1, 5 and 7 follow as immediate corollaries. Statement 1 of corollary 34 below is the most general result mentioned by Suzuki in [Suz99, p. 1594] and was also noted briefly by Vickers and Welch [VW01, p. 1090]. Theorem 33 can be viewed as a generic embedding analogue of it.

**Corollary 34.** *Do not assume AC.*

- (1) (Suzuki) *For any transitive class  $M$ , there is no nontrivial elementary embedding  $j : M \rightarrow V$ , with a critical point, that is definable with parameters in  $V$ .*
- (2) *There is no nontrivial elementary embedding  $j : V \rightarrow V$  that is definable with parameters in a set-forcing extension  $V[G]$ .*
- (3) *There is no nontrivial elementary embedding  $j : V[G] \rightarrow V$  that is definable from parameters in such a forcing extension  $V[G]$ .*
- (4) *There is no elementary embedding  $j : V \rightarrow V[G]$ , with a critical point, that is definable from parameters in such  $V[G]$ .*

*Proof.* This is a theorem scheme, asserting of each formula  $\varphi$  that it does not or is forced not to define such an elementary embedding as stated in the relevant model. Statement (1) is the special case of theorem 33 where the forcing was trivial. In statements (2) and (3), we are guaranteed the existence of a critical point by lemma 4, and so these are direct instances of theorem 33. Statement (4) follows from statement (1) applied in  $V[G]$ .  $\square$

Next, we turn to the impossibility of definable nontrivial elementary embeddings from HOD to HOD.

**Theorem 35.** *Do not assume AC. There is no nontrivial elementary embedding  $j : \text{HOD} \rightarrow \text{HOD}$  that is definable in  $V$  from parameters.*

*Proof.* This is formally a ZF theorem scheme, asserting of each formula that it cannot define such an embedding. Suppose that  $j : \text{HOD} \rightarrow \text{HOD}$  is a nontrivial elementary embedding from HOD to HOD defined from parameter  $b$  by the formula  $\varphi$ , so that  $j(x) = y$  if and only if

$V \models \varphi(x, y, b)$ . (We do not assume that  $b$  is in HOD.) Let  $\kappa$  be the critical point of  $j$ , which exists by lemma 4. Let  $\theta$  be the  $\in$ -rank of  $b$ , so that  $b \in V_{\theta+1}$ . By the Lévy reflection theorem, there is an ordinal-definable closed unbounded class  $C$  of cardinals  $\gamma$  above  $\theta$  for which the formulas  $\varphi$  and  $\exists y \varphi(x, y, z)$  are absolute between  $V_\gamma$  and  $V$ . It follows from this absoluteness that  $j \restriction \gamma \subseteq \gamma$  for any  $\gamma \in C$ . Let  $\delta$  be the  $\omega^{\text{th}}$  element of  $C$  above  $\kappa$  and  $\theta$ . In particular,  $j \restriction \delta \subseteq \delta$  and HOD believes that  $\delta$  has cofinality  $\omega$ , which is less than  $\kappa$ . These two facts imply that  $j(\delta) = \delta$ . From this, it follows that  $j((\delta^+)^{\text{HOD}}) = (\delta^+)^{\text{HOD}}$ . Thus, we have found that  $j$  has a fixed point above  $\kappa$  that is regular in HOD. Now, let  $\gamma$  be the  $(\delta^+)^{\text{HOD}}$ -th element of  $C$  above  $\kappa$  and  $\theta$ . Thus,  $\gamma \in C$  and so  $j \restriction \gamma \subseteq \gamma$ . Since  $C$  is definable in  $V$  without parameters, it follows that every initial segment of  $C$  is in HOD, and so HOD can see that  $\text{cof}(\gamma) = (\delta^+)^{\text{HOD}}$ . And since  $(\delta^+)^{\text{HOD}}$  is fixed by  $j$ , it follows that  $j(\gamma) = \gamma$ .

The point now is that this is enough to run the main stationary-partition argument, as in theorem 5. Namely, by the Ulam-Solovay theorem in HOD, there is a partition of  $(\text{Cof}_\omega \gamma)^{\text{HOD}} = \bigsqcup_{\alpha < \kappa} S_\alpha$  into stationary sets  $\vec{S} = \langle S_\alpha \mid \alpha < \kappa \rangle \in \text{HOD}$ . Let  $\vec{T} = j(\vec{S}) = \langle T_\alpha \mid \alpha < j(\kappa) \rangle$ , and let  $S^* = T_\kappa$ , which is a stationary subset of  $(\text{Cof}_\omega \gamma)^{\text{HOD}}$  in HOD. Since  $C \cap \gamma \in \text{HOD}$  and  $C \cap \gamma$  is closed and unbounded in  $\gamma$ , there is  $\beta \in S^* \cap C$ . Since  $\beta \in S^*$ , it follows that  $\beta$  has cofinality  $\omega$  in HOD and consequently  $\beta \in S_\alpha$  for some  $\alpha < \kappa$ . Since  $\beta \in C$ , however, it follows also that  $j \restriction \beta \subseteq \beta$  and consequently  $j(\beta) = \beta$ , which means that  $\beta = j(\beta) \in j(S_\alpha) = T_\alpha$ . Thus,  $\beta$  is in both  $T_\alpha$  and  $T_\kappa$ , contradicting the fact that these are disjoint, being distinct elements of the partition  $\vec{T}$ .  $\square$

Theorem 35 is generalized by theorem 37 below, which uses a different method. Note that we could have used the reflection argument involving  $C$  of this proof in several of the earlier cases, where we had wanted to find a club of closure points of  $j$ . In general, when  $j$  is merely an NGB class, not necessarily definable, we can still apply the Reflection theorem in  $\text{ZFC}(j)$  to find a closed unbounded class  $C$  of cardinals  $\gamma$  reflecting a given finite collection of statements with class parameter  $j$ , and  $C$  will be definable from  $j$ .

The proof of theorem 35 is much simpler in the case where  $j$  is definable without parameters or with ordinal parameters, for in this case one gets directly that  $j \restriction \theta \in \text{HOD}$  for every ordinal  $\theta$ , and this is enough to complete the argument. Indeed, when  $j : \text{HOD} \rightarrow \text{HOD}$  is definable in  $V$  using no parameters or using ordinal parameters, then

HOD satisfies  $\text{ZFC}(j)$  and so we have directly an instance of the Kunen inconsistency by restricting to  $\langle \text{HOD}, \in, j \rangle$ .

Since theorems 11 and 23 already rule out all class embeddings of the form  $V \rightarrow \text{HOD}$  or  $\text{HOD} \rightarrow V$ , without using AC, there is no need to consider definable embeddings of that form.

Note that we made the assumption in theorem 33 and also in corollary 34 that the embedding  $j$  has a critical point, because the situation of these results are not directly covered by lemma 4. It is however conceivable to us that a strengthened version of lemma 4 might be possible, showing that all such embeddings must have a critical point and allowing us to eliminate that assumption.

Let us conclude with a generalization of theorem 35. Using the notation  $[A]^\omega$  to denote the set of subsets of  $A$  of order type  $\omega$ , where  $A$  is a set of ordinals, we shall appeal to the Erdős-Hajnal theorem (see [Kan04]) in the following form: If  $\lambda$  is any ordinal, then there is an  $\omega$ -Jónsson function for  $\lambda$ , that is, a function  $f : [\lambda]^\omega \rightarrow \lambda$  such that for any  $A \subseteq \lambda$  of size  $\lambda$ , then  $f \restriction [A]^\omega = \lambda$ . Let  $\text{ZFC}_{+2}$  be the theory consisting of the sentences ZFC proves to be true in  $V_{\lambda+2}$ , whenever  $\lambda$  is a limit ordinal. For example, one of these sentences is the existence of an  $\omega$ -Jónsson function  $f : [\lambda]^\omega \rightarrow \lambda$ , since such a function exists in  $V_{\lambda+2}$ , if one uses a flat pairing function.

**Lemma 36.** *Suppose that  $j : M \rightarrow N$  is an elementary embedding of transitive models  $M, N \models \text{ZFC}_{+2}$ , both of height  $\lambda + 2$ , and that  $([\lambda]^\omega)^N \subseteq M$  and  $\kappa < j(\kappa)$  for some  $\kappa < \lambda$ . Then there is no set  $T \subseteq \lambda$  of size  $\lambda$  in  $N$  such that  $j(\beta) = \beta$  for all  $\beta \in T$ .*

*Proof.* Suppose that  $j : M \rightarrow N$  is elementary and that  $M$  and  $N$  are transitive models of  $\text{ZFC}_{+2}$  of height  $\lambda + 2$  and that  $\kappa < \lambda$  is the critical point of  $j$ . Note that  $j(\lambda) = \lambda$ , since this is the second largest ordinal of both  $M$  and  $N$ . Suppose toward contradiction that there is a set  $T \subseteq \lambda$  of size  $\lambda$  in  $N$  consisting of fixed points of  $j$ . By the Erdős-Hajnal theorem, there is an  $\omega$ -Jónsson function  $f : [\lambda]^\omega \rightarrow \lambda$  in  $M$ . By elementarity,  $j(f)$  is an  $\omega$ -Jónsson function  $j(f) : [\lambda]^\omega \rightarrow \lambda$  in  $N$ . Since  $T \subseteq \lambda$  has size  $\lambda$ , it follows by the  $\omega$ -Jónsson property that there is some  $s \in [T]^\omega$  in  $N$  such that  $j(f)(s) = \kappa$ , since the map is onto  $\lambda$ . Our assumption that  $([\lambda]^\omega)^N \subseteq M$  implies that  $s \in M$ . But since  $s$  has order type  $\omega$  and  $j(\beta) = \beta$  for each  $\beta \in T$ , it follows that  $j(s) = j \restriction s = s$ . Thus,  $\kappa = j(f)(s) = j(f)(j(s)) = j(f(s))$ , which would place  $\kappa$  in the range of  $j$ , which is impossible for the critical point.  $\square$

**Theorem 37.** *Do not assume AC. Suppose that  $M$  and  $N$  are transitive proper class models of ZFC with  $\text{HOD} \subseteq N$  and  $([\text{ORD}]^\omega)^N \subseteq M$ .*

*Then there is no nontrivial elementary embedding  $j : M \rightarrow N$  that is definable in  $V$  from parameters.*

*Proof.* Suppose that  $j$  is defined by the formula  $\varphi$  and parameter  $s$ , so that  $j(x) = y \iff V \models \varphi(x, y, s)$ . Let  $\kappa$  be the critical point of  $j$ , which exists by lemma 4. By the Lévy reflection theorem, there is an ordinal-definable closed unbounded class  $C$  of cardinals  $\delta$  above  $\kappa$  and the rank of  $s$  such that  $\varphi$  and  $\exists y \varphi$  both reflect from  $V$  to  $V_\delta$ . It follows as in the proof of theorem 35 that every  $\delta \in C$  has  $j \restriction \delta \subseteq \delta$ . Notice that if  $\delta$  is the  $\omega^{\text{th}}$  element of  $C$  above any ordinal—these are exactly the *simple limits* of  $C$ , that is, limit points of  $C$  that are not limits of limits of  $C$ —then  $\delta$  will have cofinality  $\omega$  in HOD and hence also in  $N$  and hence in  $M$ , from which it follows that  $j(\delta) = \delta$ . Let  $\beta_0$  be the first simple limit of  $C$ ; let  $\beta_{n+1}$  be the  $\beta_n^{\text{th}}$  simple limit of  $C$ ; and let  $\lambda = \sup_n \beta_n$ . It follows that  $\lambda \in C$  and that  $\lambda$  has cofinality  $\omega$  in HOD and hence in  $N$  and hence in  $M$ , and this implies  $j(\lambda) = \lambda$ . But also, the set of simple limits of  $C$  below  $\lambda$  has size  $\lambda$  and is in HOD and hence in  $N$ . Since these are all fixed points of  $j$ , the restriction  $j \restriction V_{\lambda+2}^M \rightarrow V_{\lambda+2}^N$  violates lemma 36, a contradiction.  $\square$

## 7. OPEN QUESTIONS

We conclude this article by mentioning the two most prominent open questions remaining in this area. Perhaps the principal open question is whether one can prove the Kunen inconsistency without using the axiom of choice.

**Question 38.** *Is it consistent without AC that  $j : V \rightarrow V$  is a nontrivial elementary embedding of the universe to itself?*

We are also naturally interested in the corresponding question for each of the generalizations of the Kunen inconsistency whose current proofs use AC. For example, in the  $\neg\text{AC}$  context can there be nontrivial elementary embeddings  $j : V[G] \rightarrow V$  or  $j : V \rightarrow V[G]$  for a set-forcing extension  $V[G]$ ?

Of course, theorem 32 and the other results of section 6 settle the case of definable embeddings  $j$  in ZF. In particular, if one understands the Kunen inconsistency solely as a ZF scheme, in the manner we described in section 1, then one might regard question 38 as settled by theorem 32. But we ask the question in the context of NGB, where the embedding  $j$  will be merely a class in NGB, not necessarily first-order definable from parameters. With this interpretation, the theorems here on definable embeddings do not answer it. An equivalent formalization of the question would inquire whether it is consistent with  $\text{ZF}(j)$ , which

allows formulas using the class  $j$  into the Separation and Replacement schemes, that  $j : V \rightarrow V$  is a nontrivial elementary embedding of the universe to itself. All of the arguments we have used in this article to refute the existence of such  $j$  have either used the stationary partition theorem, which relies on AC, or have made assumptions on the definability of  $j$ . Thus, they do not answer question 38.

A second, related open question is whether there can be a nontrivial elementary embedding from HOD to HOD.

**Question 39.** *Is it consistent that there is a nontrivial elementary embedding  $j : \text{HOD} \rightarrow \text{HOD}$ ?*

We ask the question in the NGBC or ZFC( $j$ ) contexts, although it is also sensible to drop AC here. Theorem 35 refutes without AC such  $j$  that are definable from parameters. Of course, if  $j : V \rightarrow V$  is a nontrivial elementary embedding, then so is  $j \upharpoonright \text{HOD} : \text{HOD} \rightarrow \text{HOD}$ , and so an affirmative answer to question 38 would imply an affirmative answer to question 39 in the  $\neg\text{AC}$  context. A negative answer to question 39 in the  $\neg\text{AC}$  context, in addition to implying a negative answer to question 38, would also imply by corollary 31 that there is no nontrivial elementary embedding  $j : M \rightarrow V$  whenever  $M$  is definable without parameters or with parameters fixed by  $j$ .

In addition to these two questions, of course, there are numerous others. For example, to what extent do the theorems we have mentioned about embeddings arising in set-forcing extensions also apply to class forcing? Or to certain kinds of class forcing? Or to other non-forcing extensions? To what extent do the theorems on HOD generalize to other natural definable classes? We should like to know the answers.

## REFERENCES

- [Cai] Andrés Caicedo (mathoverflow.net/users/6085). elementary embeddings. MathOverflow. <http://mathoverflow.net/questions/81568> (version: 2011-11-22).
- [Cai03] Andrés Caicedo. *Simply definable well-orderings of the reals*. dissertation, University of California at Berkeley, Berkeley, California, 2003.
- [FHR] Gunter Fuchs, Joel David Hamkins, and Jonas Reitz. Set-theoretic geology. submitted.
- [Fuc08] Gunter Fuchs. Closed maximality principles: Implications, separations and combinations. *Journal of Symbolic Logic*, 73(1):276–308, 2008.
- [Gai74] H. Gaifman. Elementary embeddings of models of set-theory and certain subtheories. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)*, pages 33–101. Amer. Math. Soc., Providence R.I., 1974.
- [GHJ] Victoria Gitman, Joel David Hamkins, and Thomas A. Johnstone. What is the theory ZFC without Powerset? submitted.



- [Jec03] Thomas Jech. *Set Theory*. Springer Monographs in Mathematics, 3rd edition, 2003.
- [Kan97] Akihiro Kanamori. *The Higher Infinite*. Springer-Verlag, 1997.
- [Kan04] Akihiro Kanamori. *The Higher Infinite, Corrected Second Edition*. Springer-Verlag, 2004.
- [Kel65] J. L. Kelley. *General Topology*. D. Van Nostrand Co., Inc., Princeton, 1965.
- [Kun71] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *Journal of Symbolic Logic*, 36:407–413, 1971.
- [McA74] Kenneth McAloon. On the sequence of models  $\text{HOD}_n$ . *Fundamenta Mathematicae*, 82:85–93, 1974.
- [NZ98] Itay Neeman and Jindřich Zapletal. Proper forcing and absoluteness in  $L(\mathbb{R})$ . *Math. Uni. Carolina*, 39:281–301, 1998.
- [NZ01] Itay Neeman and Jindřich Zapletal. Proper forcing and  $L(\mathbb{R})$ . *Journal of Symbolic Logic*, 66:801–810, 2001.
- [Suz98] Akira Suzuki. Non-existence of generic elementary embeddings into the ground model. *Tsukuba J. Math.*, 22(2):343–347, 1998.
- [Suz99] Akira Suzuki. No elementary embedding from  $V$  into  $V$  is definable from parameters. *J. Symbolic Logic*, 64(4):1591–1594, 1999.
- [VW01] J. Vickers and P. D. Welch. On elementary embeddings from an inner model to the universe. *J. Symbolic Logic*, 66(3):1090–1116, 2001.
- [Zad83] Włodzimierz Zadrozny. Iterating ordinal definability. *Ann. Pure Appl. Logic*, 24(3):263–310, 1983.
- [Zap96] Jindřich Zapletal. A new proof of Kunen’s inconsistency. *Proc. Amer. Math. Soc.*, 124(7):2203–2204, 1996.

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